

On the determination of the Singer transfer

Sum Nguyen*

Department of Mathematics, Quy Nhon University

Received 23 October 2017; accepted 24 January 2018

Abstract:

Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \dots, x_k]$ with the degree of each generator x_i being 1, where \mathbb{F}_2 denote the prime field of two elements, and let GL_k be the general linear group over \mathbb{F}_2 which acts regularly on P_k .

We study the algebraic transfer constructed by Singer [1] using the technique of the Peterson hit problem. This transfer is a homomorphism from the homology of the mod-2 Steenrod algebra \mathcal{A} , $Tor_{k,k+d}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, to the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of all the GL_k -invariant classes of degree d .

In this paper, by using the results on the Peterson hit problem we present the proof of the fact that the Singer algebraic transfer is an isomorphism for $k \leq 3$. This result has been proved by Singer in [1] for $k \leq 2$ and by Boardman in [2] for $k = 3$. We show that the fourth Singer transfer is also an isomorphism in certain internal degrees. This result is new and it is different from the ones of Bruner, Ha and Hung [3], Chon and Ha [4], Ha [5], Hung and Quynh [6], Nam [7].

Keywords: algebraic transfer, polynomial algebra, steenrod algebra.

Classification number: 1.1

1. Introduction

Denote by $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$ the polynomial algebra over the field of two elements, \mathbb{F}_2 , in k generators x_1, x_2, \dots, x_k , each of degree 1. This algebra arises as the cohomology with coefficients in \mathbb{F}_2 of an elementary abelian 2-group of rank k . Therefore, P_k is a module over the mod-2 Steenrod algebra, \mathcal{A} . The action of \mathcal{A} on P_n is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula $Sq^k(fg) = \sum_{i=0}^k Sq^i(f)Sq^{k-i}(g)$, for $f, g \in P_k$ [8].

The Peterson hit problem is to find a minimal generating set for P_k regarded as a module over the mod-2 Steenrod algebra. Equivalently, this problem is to find a vector space basis for $QP_k := \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ in each degree d . Such a basis may be represented by a list of monomials of degree d . It is completely determined for $k \leq 4$, unknown in general.

Let GL_k be the general linear group over the field \mathbb{F}_2 . This group acts naturally on P_k by matrix substitution. Since the two actions of \mathcal{A} and GL_k upon P_k commute with each other, there is an inherited action of GL_k on QP_k .

Denote by $(P_k)_d$ the subspace of P_k consisting of all the homogeneous polynomials of degree d in P_k and by $(QP_k)_d$ the subspace of QP_k consisting of all the classes represented by the elements in $(P_k)_d$. In [1], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : Tor_{k,k+d}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_k)_d^{GL_k}$$

from the homology of the Steenrod algebra to the subspace of $(QP_k)_d$ consisting of all the GL_k -invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $Tor_{k,k+d}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. This transfer was studied by Boardman [2], Bruner, Ha and Hung [3], Ha [5], Hung [9], Chon and Ha [4, 10, 11], Minami [12], Nam [7], Hung and Quynh [6], the present author [13] and others.

Singer showed in [1] that φ_k is an isomorphism for $k = 1, 2$. Boardman showed in [2] that φ_3 is also an isomorphism. However, for any $k \geq 4$, φ_k is not a monomorphism in infinitely many degrees (see Singer [1], Bruner, Ha and Hung [3], Hung [9]). Singer made a conjecture in [1] that the algebraic transfer φ_k is an epimorphism for any $k \geq 0$. This conjecture is true for $k \leq 3$. It can be verified for $k = 4$ by using the results in [14, 15]. The conjecture for $k \geq 5$ is an open problem.

*Email: nguyensum@qnu.edu.vn.

In this paper, by using the results on the Peterson hit problem we present the proof of the fact that the Singer algebraic transfer is an isomorphism for $k \leq 3$. Recall that this result has been proved by Singer in [1] for $k \leq 2$ and by Boardman in [2] for $k = 3$. To prove this result, Boardman [2] computed the space $QP_3^{GL_3}$ by using a basis consisting of the all the classes represented by certain polynomials in P_3 . We also compute this space, however we use the admissible monomial basis for QP_3 that is different from the one of Boardman in [2]. By applying this technique for $k = 4$, we show that the fourth Singer transfer is also an isomorphism in certain internal degrees. This result is new and it is different from the ones of Bruner, Ha and Hung [3], Chon and Ha [4], Ha [5], Hung and Quynh [6], Nam [7]. In those works it is shown only that the fourth Singer transfer detects certain families of elements in $\text{Ext}_{\mathcal{A}}^{4,*}(\mathbb{F}_2, \mathbb{F}_2)$, and fails to detect others.

This paper is organized as follows. In Section 2, we recall some needed information on the lambda algebra and the Singer algebraic transfer. In Sections 3, we present the determination of the algebraic transfer for $k \leq 3$. Finally, in Section 4, we show that the fourth Singer transfer is an isomorphism in certain internal degrees.

2. The Singer algebraic transfer and the lambda algebra

First of all, we briefly recall the definition of the Singer transfer. Let \widehat{P}_1 be the submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$ spanned by all powers x_1^i with $i \geq -1$. The usual \mathcal{A} -action on $P_1 = \mathbb{F}_2[x_1]$ is canonically extended to an \mathcal{A} -action on $\mathbb{F}_2[x_1, x_1^{-1}]$ (see Singer [1]). \widehat{P}_1 is an \mathcal{A} -submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$. The inclusion $P_1 \subset \widehat{P}_1$ gives rise to a short exact sequence of \mathcal{A} -modules:

$$0 \rightarrow P_1 \rightarrow \widehat{P}_1 \rightarrow \Sigma^{-1}\mathbb{F}_2 \rightarrow 0.$$

Let e_1 be the corresponding element in $\text{Ext}_{\mathcal{A}}^1(\Sigma^{-1}\mathbb{F}_2, P_1)$. By using the cross and Yoneda products, Singer set

$$e_k = (e_1 \times P_{k-1}) \circ (e_1 \times P_{k-2}) \circ \dots \circ (e_1 \times P_1) \circ e_1 \in \text{Ext}_{\mathcal{A}}^k(\Sigma^{-k}\mathbb{F}_2, P_k).$$

Then, he defined $\hat{\varphi}_k : \text{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \rightarrow \text{Tor}_0^{\mathcal{A}}(\mathbb{F}_2, P_k) = QP_k$ by $\hat{\varphi}_k(z) = e_k \cap z$. Its image is a submodule of $(QP_k)^{GL_k}$. So, $\hat{\varphi}_k$ induces the homomorphism

$$\varphi_k : \text{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \rightarrow QP_k^{GL_k}.$$

Denote by $(P_k)^*$ the dual of P_k and by $P((P_k)^*)$ the primitive subspace consisting of all elements in $(P_k)^*$ that are annihilated by every positive degree operations in the mod-2 Steenrod algebra. The dual of φ_k :

$$\text{Tr}_k := (\varphi_k)^* : \mathbb{F}_2 \otimes_{GL_k} P((P_k)^*) \rightarrow \text{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$$

is also called the k -th Singer transfer.

The algebra $\text{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is described in terms of the mod-2 lambda algebra Λ (see [16]). Recall that Λ is a bigraded differential algebra over \mathbb{F}_2 generated by $\lambda_j \in \Lambda^{1,j}, j \geq 0$, with the relations

$$\lambda_j \lambda_{2j+1+m} = \sum_{\nu \geq 0} \binom{m - \nu - 1}{\nu} \lambda_{j+m-\nu} \lambda_{2j+1+\nu}, \tag{2.1}$$

for $m \geq 0$ and the differential

$$\delta(\lambda_i) = \sum_{\nu \geq 0} \binom{i - \nu - 1}{\nu + 1} \lambda_{i-\nu-1} \lambda_{\nu}, \tag{2.2}$$

for $i > 0$, $\delta(\lambda_0) = 0$ and that $H^{k,d}(\Lambda, \delta) = \text{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$.

For example, the elements $\lambda_{2^i-1} \in \Lambda^{1,2^i-1}, i \geq 0$, and $\bar{d}_0 = \lambda_6 \lambda_2 \lambda_3^2 + \lambda_4^2 \lambda_3^2 + \lambda_2 \lambda_4 \lambda_5 \lambda_3 + \lambda_1 \lambda_5 \lambda_1 \lambda_7 \in \Lambda^{4,14}$ are the cycles in the lambda algebra Λ . So, $h_i = [\lambda_{2^i-1}]$ and $d_0 = [\bar{d}_0]$ are the elements in $\text{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. Note that h_i is the Adams element in $\text{Ext}_{\mathcal{A}}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$.

There is a homomorphism $\widetilde{Sq}^0 : \Lambda \rightarrow \Lambda$ determined by

$$\widetilde{Sq}^0(\lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}) = \lambda_{2j_1+1} \lambda_{2j_2+1} \dots \lambda_{2j_k+1}, k \geq 0.$$

This homomorphism respects the relations in ((2.1)) and commutes the differential in ((2.2)). Therefore, it induces a homomorphism

$$Sq^0 : \text{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) = H^{k,d}(\Lambda) \rightarrow H^{k,k+2d}(\Lambda) = \text{Ext}_{\mathcal{A}}^{k,2k+2d}(\mathbb{F}_2, \mathbb{F}_2).$$

A family $\{a_i : i \geq 0\}$ of elements in $\text{Ext}_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2)$ is called a Sq^0 -family if $a_i = (Sq^0)^i(a_0)$ for every $i \geq 0$. It is well known that $\text{Ext}_{\mathcal{A}}^{3,3+*}(\mathbb{F}_2, \mathbb{F}_2)$ contains the Sq^0 -family of indecomposable elements $\{c_i\}$ and $\text{Ext}_{\mathcal{A}}^{4,4+*}(\mathbb{F}_2, \mathbb{F}_2)$

contains seven Sq^0 -families of indecomposable elements, namely $\{d_i\}, \{e_i\}, \{f_i\}, \{g_{i+1}\}, \{p_i\}, \{D_3(i)\}$, and $\{p'_i\}$. Note that $\{h_i\}$ is also a Sq^0 -family in $\text{Ext}_{\mathcal{A}}^{1,1+*}(\mathbb{F}_2, \mathbb{F}_2)$.

The algebra $\{\text{Ext}_{\mathcal{A}}^{k,k+*}(\mathbb{F}_2, \mathbb{F}_2) | k \geq 0\}$ has been explicitly computed by Adem [17] for $k = 1$, by Adams [18] and Wall [19] for $k = 2$, by Adams [18] and Wang [20] for $k = 3$ and by Lin [21] for $k = 4$.

Theorem 2.1 (see 17-21)

i) *The algebra $\{\text{Ext}_{\mathcal{A}}^{k,k+*}(\mathbb{F}_2, \mathbb{F}_2) | k \geq 0\}$ for $k \leq 3$ is generated by h_i and c_i for $i \geq 0$ and subject only to the relations $h_i h_{i+1} = 0, h_i h_{i+2}^2 = 0$ and $h_i^3 = h_{i-1}^2 h_{i+1}$. In particular, $\{c_i : i \geq 0\}$ is an \mathbb{F}_2 -basis for the indecomposable elements in $\text{Ext}_{\mathcal{A}}^{3,3+*}(\mathbb{F}_2, \mathbb{F}_2)$.*

ii) *The algebra $\{\text{Ext}_{\mathcal{A}}^{k,k+*}(\mathbb{F}_2, \mathbb{F}_2) | k \geq 0\}$ for $k \leq 4$ is generated by $h_i, c_i, d_i, e_i, f_i, g_{i+1}, p_i, D_3(i)$ and p'_i for $i \geq 0$ and subject to the relations in i) together with the relations $h_i^2 h_{i+3}^2 = 0, h_j c_i = 0$ for $j = i - 1, i, i + 2$ and $i + 3$. Furthermore, the set of the elements $d_i, e_i, f_i, g_{i+1}, p_i, D_3(i)$ and p'_i , for $i \geq 0$, is an \mathbb{F}_2 -basis for the indecomposable elements in $\text{Ext}_{\mathcal{A}}^{4,4+*}(\mathbb{F}_2, \mathbb{F}_2)$.*

It is well known that the dual of P_k is the divided power algebra generated by a_1, a_2, \dots, a_k :

$$(P_k)^* = \Gamma(a_1, a_2, \dots, a_k)$$

where $a_j^{(i)}$ is dual to $x_j^i \in P_k$ with respect to the basis of P_k consisting of all monomials in x_1, x_2, \dots, x_k and $a_j = a_j^{(1)}$. The graded vector space $\{(P_k)^* | k \geq 0\}$ is an algebra with a multiplication defined by

$$(a_1^{(i_1)} \dots a_k^{(i_k)})(a_1^{(i_{k+1})} \dots a_m^{(i_{k+m})}) = a_1^{(i_1)} \dots a_k^{(i_k)} a_{k+1}^{(i_{k+1})} \dots a_{k+m}^{(i_{k+m})} \in (P_{k+m})^*,$$

for any $a_1^{(i_1)} \dots a_k^{(i_k)} \in (P_k)^*$ and $a_1^{(i_{k+1})} \dots a_m^{(i_{k+m})} \in (P_m)^*$. In [10], Chon and Ha defined a homomorphism of algebras

$$\phi = \{\phi_k | k \geq 0\} : \{(P_k)^* | k \geq 0\} \longrightarrow \{\Lambda^{k,*} | k \geq 0\} = \Lambda,$$

which induces the Singer transfer. Here, the homomorphism $\phi_k : (P_k)^* \rightarrow \Lambda^{k,*}$ is defined by the following inductive formula:

$$\phi_k(a^{(I,t)}) = \begin{cases} \lambda_t, & \text{if } k - 1 = \ell(I) = 0, \\ \sum_{i \geq t} \phi_{k-1}(Sq^{i-t} a^I) \lambda_i, & \text{if } k - 1 = \ell(I) > 0, \end{cases}$$

for any $a^{(I,t)} = a_1^{(i_1)} a_2^{(i_2)} \dots a_{k-1}^{(i_{k-1})} a_k^{(t)} \in (P_k)^*$ and $I = (i_1, i_2, \dots, i_{k-1})$.

Theorem 2.2 (see Chon and Ha [10]) *If $b \in P((P_k)^*)$, then $\phi_k(b)$ is a cycle in the lambda algebra Λ and $Tr_k([b]) = [\phi_k(b)]$.*

Note that this theorem is a dual version of the one in Hung [22].

We end this section by recalling some results on Kameko's homomorphism and the generators of the general linear group GL_k .

One of the main tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}_*^0 : QP_k \rightarrow QP_k$. This homomorphism is induced by the \mathbb{F}_2 -linear map $\psi : P_k \rightarrow P_k$, given by

$$\psi(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \dots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. Note that ψ is not an \mathcal{A} -homomorphism. However, $\psi Sq^{2t} = Sq^t \psi$, and $\psi Sq^{2t+1} = 0$ for any non-negative integer t .

For a positive integer n , by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1 \leq i \leq r} (2^{u_i} - 1)$, where $u_i > 0$.

Theorem 2.3 (Kameko [23]) *Let m be a positive integer. If $\mu(2m+k) = k$, then $(\widetilde{Sq}_*)^0_m : (QP_k)_{2m+k} \rightarrow (QP_k)_m$ is an isomorphism of the GL_k -modules.*

Definition 2.4 *Let f, g be two polynomials of the same degree in P_k . Then, $f \equiv g$ if and only if $f - g \in \mathcal{A}^+ P_k$. If $f \equiv 0$, then f is called hit.*

For $1 \leq i \leq k$, define the \mathcal{A} -homomorphism $\rho_i : P_k \rightarrow P_k$, which is determined by $\rho_i(x_i) = x_{i+1}, \rho_i(x_{i+1}) = x_i, \rho_i(x_j) = x_j$ for $j \neq i, i + 1, 1 \leq i < k$, and $\rho_k(x_1) = x_1 + x_2, \rho_k(x_j) = x_j$ for $j > 1$.

It is easy to see that the general linear group GL_k is generated by the matrices associated with $\rho_i, 1 \leq i \leq k$, and the symmetric group Σ_k is generated by the ones associated with $\rho_i, 1 \leq i < k$. So, a class $[f]$ represented by a homogeneous polynomial $f \in P_k$ is an GL_k -invariant if and only if $\rho_i(f) \equiv f$ for $1 \leq i \leq k$. It is an Σ_k -invariant if and only if $\rho_i(f) \equiv f$ for $1 \leq i < k$.

3. Determination of Tr_k for $k \leq 3$

3.1. Determination of Tr_k for $k \leq 2$

In this subsection, we present the proof of the following.

Theorem 3.1.1 (Singer [1]) *The algebraic transfer Tr_k is an isomorphism for $k \leq 2$.*

It is well-known that

$$(QP_1)_n^{GL_1} = (QP_1)_n = \begin{cases} \langle [x^{2^u-1}] \rangle, & \text{if } n = 2^u - 1, u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

According to Theorem 2.1, we have

$$\text{Ext}_{\mathcal{A}}^{1,t+1}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_u \rangle, & \text{if } t = 2^u - 1, u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(P_1)^* = \Gamma(a)$ and $a^{(2^u-1)} \in P((P_1)^*)$, $\phi_1(a^{(2^u-1)}) = \lambda_{2^u-1}$ is a cycle in $\Lambda^{1,*}$. Using Theorem 2.2, we get $Tr_1([a^{(2^u-1)}]) = [\phi_1(a^{(2^u-1)})] = [\lambda_{2^u-1}] = h_u, \forall u \geq 0$. So, Tr_1 is a isomorphism.

Now, we present the proof of this theorem for $k = 2$ by computing the space $(QP_2)^{GL_2}$. From a result of Wood [24], we need only to compute this space in the degree $n = 2^{s+t} + 2^s - 2$ with s, t non-negative integers.

First, we consider the degree $n = 2^{s+1} - 2$ with $s \geq 0$. Since the iterated Kameko homomorphism $(\widetilde{Sq}_*^0)^s : (QP_2)_n \rightarrow (QP_2)_0$ is a isomorphism of GL_2 -modules and $(QP_2)_0^{GL_2} = \langle 1 \rangle$, hence $(QP_2)_n^{GL_2} = \langle [p_{2,s}] \rangle$ with $p_{2,s} := (x_1 x_2)^{2^s-1}$.

Next, we compute $(QP_2)_n^{GL_2}$ with $n = 2^{s+1} + 2^s - 2, s \geq 0$. Since the iterated Kameko homomorphism $(\widetilde{Sq}_*^0)^s : (QP_2)_n \rightarrow (QP_2)_1$ is a isomorphism of GL_2 -modules, we need only to compute $(QP_2)_1^{GL_2}$.

According to Peterson [25], $(QP_2)_n$ is the vector space of dimension 2 with a basis consisting of 2 classes represented by the following monomials:

$$v_{s,1} = x_1^{2^s-1} x_2^{2^{s+1}-1}, v_{s,2} = x_1^{2^{s+1}-1} x_2^{2^s-1}.$$

In particular, $v_{0,1} = x_2, v_{0,2} = x_1$. Suppose $\theta = a_1 v_1 + a_2 v_2 = a_1 x_2 + a_2 x_1 \in (QP_2)_1^{GL_2}$ with $a_1, a_2 \in \mathbb{F}_2$. Then $\rho_1(\theta) = a_1 v_2 + a_2 v_1 \equiv \theta$. So, we get $a_1 = a_2$. Since $\rho_2(\theta) \equiv a_1 v_1 + a_2(v_1 + v_2) \equiv \theta$, we obtain $a_1 = a_2 = 0$. Hence, $(QP_2)_1^{GL_2} = 0$ and $(QP_2)_n^{GL_2} = 0$.

Now, we consider the degree $n = 2^{s+t} + 2^s - 2$ with s, t non-negative integers, $t \geq 2$. Since $(\widetilde{Sq}_*^0)^s : (QP_2)_n \rightarrow (QP_2)_{2^t-1}$ is a isomorphism of GL_2 -modules, we need only to compute $(QP_2)_{2^t-1}^{GL_2}$. According to Peterson [25], $(QP_2)_{2^t-1}$ is the vector space of dimension 3 with a basis consisting of 3 classes represented by the following monomials:

$$u_{t,1} = x_1^{2^t-1}, u_{t,2} = x_2^{2^t-1}, u_{t,3} = x_1 x_2^{2^t-2}.$$

Suppose $\theta_t = a_1 u_{t,1} + a_2 u_{t,2} + a_3 u_{t,3}$ with $a_1, a_2, a_3 \in \mathbb{F}_2$ and $[\theta_t] \in (QP_2)_{2^t-1}^{GL_2}$. By a simple computation, we have $\rho_1(\theta_t) = a_1 u_{t,2} + a_2 u_{t,1} + a_3 u_{t,3} \equiv \theta_t$, hence $a_1 = a_2 = a$. Then, $\rho_2(\theta_t) \equiv a(u_{t,1} + u_{t,2}) + a u_{t,2} + a_3(u_{t,2} + u_{t,3}) \equiv \theta_t$. So, we get $a_3 = a$. Hence, $\theta_t = a p_{2,0,t}$ with $p_{2,0,t} = u_{t,1} + u_{t,2} + u_{t,3}$ and

$$(QP_2)_n^{GL_2} = \langle [\psi^s(p_{2,0,t})] \rangle.$$

Combining the above results, we obtain

Proposition 3.1.2 *Let n be a non-negative integer. We have*

$$(QP_2)_n^{GL_2} = \begin{cases} \langle [p_{2,s}] \rangle, & \text{if } n = 2^{s+1} - 2, s \geq 0 \\ \langle [p_{2,s,t}] \rangle, & \text{if } n = 2^{s+t} + 2^s - 2, s \geq 0, t \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $p_{2,s,t} = \psi^s(p_{2,0,t})$.

Recall that $(P_2)^* = \Gamma(a_1, a_2)$. For any $s, t \geq 0$, we set

$$q_{2,s,t} := a_1^{(2^s-1)} a_2^{(2^{s+t}-1)} \in P((P_2)_{2^{s+t}+2^s-2}^*).$$

Since $\langle q_{2,s,0}, p_{2,s} \rangle = 1$ and $\langle q_{2,s,t}, p_{2,s,t} \rangle = 1$ for every $s \geq 0, t \geq 2$, from Proposition 3.1.2, we get the following.

Proposition 3.1.3 For n a non-negative integer, we obtain

$$\mathbb{F}_2 \otimes_{GL_2} P((P_2)_n^*) = \begin{cases} \langle [q_{2,s,0}] \rangle, & \text{if } n = 2^{s+1} - 2, s \geq 0 \\ \langle [q_{2,s,t}] \rangle, & \text{if } n = 2^{s+t} + 2^s - 2, s \geq 0, t \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\phi_2(q_{2,s,t}) = \lambda_{2^s-1} \lambda_{2^{s+t}-1}$ is a cycle in $\Lambda^{2,*}$. Applying Theorem 2.2, we get

$$Tr_2([q_{2,s,t}]) = [\phi_2(q_{2,s,t})] = [\lambda_{2^s-1} \lambda_{2^{s+t}-1}] = h_s h_{s+t}.$$

Since $h_s h_{s+1} = 0$, applying Theorem 2.1, we have

$$\text{Ext}_{\mathcal{A}}^{2,m}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_s^2 \rangle, & \text{if } m = 2^{s+1}, \text{ with } s \geq 0, \\ \langle h_s h_{s+t} \rangle, & \text{if } m = 2^{s+t} + 2^s, \text{ with } s \geq 0, t \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.1.1 is completely proved.

3.2. Determination of Tr_3

In this subsection, we present the proof of the following.

Theorem 3.2.1 (Boardman [2]) *The third Singer algebraic transfer*

$$Tr_3 : \mathbb{F}_2 \otimes_{GL_3} P((P_3)^*) \longrightarrow \text{Ext}_{\mathcal{A}}^{3,*+3}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism.

To prove this theorem, Boardman [2] computed the space $QP_3^{GL_3}$ by using a basis consisting of the all the classes represented by certain polynomials in P_3 . It is difficult to use his method for $k = 4$, where there are 315 polynomials instead of 21. We also compute this space, however we use the admissible monomial basis for QP_3 that is different from the one of Boardman in [2]. Our approach can be apply for $k = 4$ by using the admissible monomial basis for QP_4 which is given in [14, 15].

From a result of Wood [24], we need only to compute $QP_3^{GL_3}$ in the degree n with $\mu(n) \leq 3$.

3.2.1. The case $n = 2^{t+1} - 2$

According to Kameko [23], $(QP_3)_n$ is a vector space with a basis consisting of all the classes represented by the following monomials:

$$\begin{aligned} v_{t,1} &= x_2^{2^t-1} x_3^{2^t-1}, v_{t,2} = x_1^{2^t-1} x_3^{2^t-1}, v_{t,3} = x_1^{2^t-1} x_2^{2^t-1}, \text{ for } t \geq 1, \\ v_{t,4} &= x_1 x_2^{2^t-2} x_3^{2^t-1}, v_{t,5} = x_1 x_2^{2^t-1} x_3^{2^t-2}, v_{t,6} = x_1^{2^t-1} x_2 x_3^{2^t-2}, \text{ for } t \geq 2, \\ v_{t,7} &= x_1^3 x_2^{2^t-3} x_3^{2^t-2}, \text{ for } t \geq 3 \end{aligned}$$

Set $p_{3,t} = \sum_{i=1}^7 v_{t,i}$, with $t \geq 3$. By a direct computation, we have

Proposition 3.2.2 For any non-negative integer t , we have

$$(QP_3)_{2^{t+1}-2}^{GL_3} = \begin{cases} \langle 1 \rangle, & \text{if } t = 0, \\ 0, & \text{if } t = 1, 2, \\ \langle [p_{3,t}] \rangle, & \text{if } t \geq 3. \end{cases}$$

Recall that $(P_3)^* = \Gamma(a_1, a_2, a_3)$. We set

$$q_{3,t} = a_1^{(0)} a_2^{(2^t-1)} a_3^{(2^t-1)} \in P((P_3)_{2^{t+1}-2}^*).$$

Since $\langle p_{3,t}, q_{3,t} \rangle = 1$, we get

$$\mathbb{F}_2 \otimes_{GL_3} P((P_3)_{2^{t+1}-2}^*) = \begin{cases} \langle [1] \rangle, & \text{if } t = 0 \\ 0, & \text{if } t = 1, 2, \\ \langle [q_{3,t}] \rangle, & \text{if } t \geq 3. \end{cases}$$

It is easy to see that $\phi_3(q_{3,t}) = \lambda_0 \lambda_{2^t-1}^2$ is a cycle in $\Lambda^{3,*}$. By Theorem 2.2, we have

$$Tr_3([q_{3,t}]) = [\phi_3(q_{3,t})] = [\lambda_0 \lambda_{2^t-1}^2] = h_0 h_t^2.$$

According to Theorem 2.1, we have

$$\text{Ext}_{\mathcal{A}}^{3,2^{t+1}+1}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_0 h_t^2 \rangle.$$

Since $h_0 h_1 = 0$ and $h_0 h_2^2 = 0$, from the above equalities we see that Theorem 3.2.1 is true in this case.

3.2.2. The case $n = 2^{t+u} + 2^u - 3$

If $u > 1$ then $\mu(n) = 3$, hence the iterated Kameko homomorphism

$$(\widetilde{Sq}_*^0)^{u-1} : (QP_3)_{2^{t+u}+2^u-3} \rightarrow (QP_3)_{2^{t+1}-1}$$

is also an isomorphism GL_3 -modules. Hence, we need only to compute $(QP_3)_{2^{t+1}-1}^{GL_3}$. According to Kameko [23], $(QP_3)_n$ is a vector space with a basis consisting of all the classes represented by the following monomials:

$$\begin{aligned} u_{t,1} &= x_3^{2^{t+1}-1}, \quad u_{t,2} = x_2^{2^{t+1}-1}, \quad u_{t,3} = x_1^{2^{t+1}-1}, \quad \text{for } t \geq 0, \\ u_{t,4} &= x_2 x_3^{2^{t+1}-2}, \quad u_{t,5} = x_1 x_3^{2^{t+1}-2}, \quad u_{t,6} = x_1 x_2^{2^{t+1}-2}, \quad \text{for } t \geq 1, \\ u_{t,7} &= x_1 x_2 x_3, \quad \text{for } t = 1, \\ u_{t,7} &= x_1 x_2^2 x_3^{2^{t+1}-4}, \quad u_{t,8} = x_1 x_2^{2^t-1} x_3^{2^t-1}, \\ u_{t,9} &= x_1^{2^t-1} x_2 x_3^{2^t-1}, \quad u_{t,10} = x_1^{2^t-1} x_2^{2^t-1} x_3, \quad \text{for } t \geq 2, \\ u_{t,11} &= x_1^3 x_2^{2^t-3} x_3^{2^t-1}, \quad u_{t,12} = x_1^3 x_2^{2^t-1} x_3^{2^t-3}, \quad u_{t,13} = x_1^{2^t-1} x_2^3 x_3^{2^t-3}, \quad \text{for } t \geq 3, \\ u_{t,14} &= x_1^7 x_2^{2^t-5} x_3^{2^t-3}, \quad \text{for } t \geq 4. \end{aligned}$$

Set $p_{3,t,1} = \sum_{i=1}^7 u_{t,i}$ for $t \geq 1$ and $\bar{p}_{3,t,1} = \sum_{j=7}^{14} u_{t,j}$ for $t \geq 4$. By a direct computation we have

Proposition 3.2.3 For any integers $t \geq 0, u > 0$, we have

$$(QP_3)_{2^{t+u}+2^u-3}^{GL_3} = \begin{cases} 0, & \text{if } t = 0, \\ \langle [p_{3,t,u}] \rangle, & \text{if } 1 \leq t \leq 3, \\ \langle [p_{3,t,u}], [\bar{p}_{3,t,u}] \rangle, & \text{if } t \geq 4, \end{cases}$$

where $p_{3,t,u} = \psi^{u-1}(p_{3,t,1}), \bar{p}_{3,t,u} = \psi^{u-1}(\bar{p}_{3,t,1})$.

We set

$$q_{3,t,u} = a_1^{(2^{u-1}-1)} a_2^{(2^{u-1}-1)} a_3^{(2^{t+u}-1)}, \quad \bar{q}_{3,t,u} = a_1^{(2^u-1)} a_2^{(2^{t+u-1}-1)} a_3^{(2^{t+u-1}-1)}.$$

It is easy to see that $q_{3,t,u}, \bar{q}_{3,t,u} \in P((P_3)_{2^{t+1}-2}^*)$ and

$$\begin{aligned} \langle p_{3,t,u}, q_{3,t,u} \rangle &= 1, \quad \langle p_{3,t,u}, \bar{q}_{3,t,u} \rangle = 0, \\ \langle \bar{p}_{3,t,u}, q_{3,t,u} \rangle &= 0, \quad \langle \bar{p}_{3,t,u}, \bar{q}_{3,t,u} \rangle = 1. \end{aligned}$$

So, we get

$$\mathbb{F}_2 \otimes_{GL_3} P((P_3)_{2^{t+u}+2^u-3}^*) = \begin{cases} 0, & \text{if } t = 0, \\ \langle [q_{3,t,u}] \rangle, & \text{if } 1 \leq t \leq 3, \\ \langle [q_{3,t,u}], [\bar{q}_{3,t,u}] \rangle, & \text{if } t \geq 4. \end{cases}$$

By applying Theorem 2.2, we have

$$\begin{aligned} \phi_3(q_{3,t,u}) &= \lambda_{2^{u-1}-1}^2 \lambda_{2^{t+u}-1}, \\ \phi_3(\bar{q}_{3,t,u}) &= \lambda_{2^u-1} \lambda_{2^{t+u-1}-1}^2 \end{aligned}$$

are the cycles in $\Lambda^{3,*}$. So, we obtain

$$\begin{aligned} Tr_3([q_{3,t,u}]) &= [\phi_3(q_{3,t,u})] = [\lambda_{2^{u-1}-1}^2 \lambda_{2^{t+u}-1}] = h_{u-1}^2 h_{t+u}, \\ Tr_3([\bar{q}_{3,t,u}]) &= [\phi_3(\bar{q}_{3,t,u})] = [\lambda_{2^u-1} \lambda_{2^{t+u-1}-1}^2] = h_u h_{t+u-1}^2. \end{aligned}$$

According to Theorem 2.1, we have

$$\text{Ext}_{\mathcal{A}}^{3,2^{t+u}+2^u}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_u h_{t+u-1}^2, h_{u-1}^2 h_{t+u} \rangle.$$

If $t = 0$ then $h_u h_{u-1}^2 = h_u^2 h_{u-1} = 0$. If $t = 1$ then $h_u h_{t+u-1}^2 = h_u^3 = h_{u-1}^2 h_{u+1} = h_{u-1}^2 h_{t+u}$. If $t = 2$ then $h_u h_{t+u-1}^2 = h_u h_{u+1}^2 = 0$. If $t = 3$ then $h_u h_{t+u-1}^2 = h_u h_{u+2}^2 = 0$. Hence, from the above equalities we can easily see that Theorem 3.2.1 is true in this case.

3.2.3. The case $n = 2^{s+u+1} + 2^{u+1} + 2^u - 3$

If $u > 0$ then $\mu(n) = 3$, hence the iterated Kameko homomorphism

$$(\widetilde{Sq}_*^0)^u : (QP_3)_{2^{s+u}+2^u-3} \rightarrow (QP_3)_{2^{s+1}}$$

is also an isomorphism of GL_3 -modules. Hence, we need only to compute $(QP_3)_{2^{s+1}}^{GL_3}$.

According to Kameko [23], $(QP_3)_{2^{s+1}}$ is a vector space with a basis consisting of all the classes represented by the following monomials:

$$\begin{aligned} v_{s,1} &= x_2 x_3^{2^{s+1}-1}, \quad v_{s,2} = x_2^{2^{s+1}-1} x_3, \quad v_{s,3} = x_1 x_3^{2^{s+1}-1}, \\ v_{s,4} &= x_1 x_2^{2^{s+1}-1}, \quad v_{s,5} = x_1^{2^{s+1}-1} x_3, \quad v_{s,6} = x_1^{2^{s+1}-1} x_2, \quad \text{for } s \geq 1, \\ v_{1,7} &= x_1 x_2 x_3^2, \quad v_{1,8} = x_1 x_2^2 x_3, \quad \text{for } s = 1, \\ v_{s,7} &= x_2^3 x_3^{2^{s+1}-3}, \quad v_{s,8} = x_1^3 x_3^{2^{s+1}-3}, \quad v_{s,9} = x_1^3 x_2^{2^{s+1}-3}, \\ v_{s,10} &= x_1 x_2 x_2^{2^{s+1}-2}, \quad v_{s,11} = x_1 x_2^{2^{s+1}-2} x_3, \quad v_{s,12} = x_1 x_2^2 x_3^{2^{s+1}-3}, \\ v_{s,13} &= x_1 x_2^3 x_3^{2^{s+1}-4}, \quad v_{s,14} = x_1^3 x_2 x_3^{2^{s+1}-4} \quad \text{for } s \geq 2 \\ v_{15} &= x_1^3 x_2^4 x_3, \quad \text{for } s = 2. \end{aligned}$$

Set $\bar{p}_0 = v_{2,10} + v_{2,11} + v_{2,14} + v_{2,15}$. By a direct computation, we have

Proposition 3.2.4 For any integers $s > 0, u \geq 0$ and $n = 2^{s+u+1} + 2^{u+1} + 2^u - 3$, we have

$$(QP_3)_n^{GL_3} = \begin{cases} \langle [\psi^u(\bar{p}_0)] \rangle, & \text{if } s = 2, \\ 0, & \text{if } s \neq 2. \end{cases}$$

We set

$$\begin{aligned} \bar{c}_u &= a_1^{(3.2^u-1)} a_2^{(4.2^u-1)} a_3^{(4.2^u-1)} + a_1^{(2.2^u-1)} a_2^{(5.2^u-1)} a_3^{(4.2^u-1)} \\ &\quad + a_1^{(2.2^u-1)} a_2^{(3.2^u-1)} a_3^{(6.2^u-1)} + a_1^{(2.2^u-1)} a_2^{(2.2^u-1)} a_3^{(7.2^u-1)} \end{aligned}$$

is an element in $(P_3)^* = \Gamma(a_1, a_2, a_3)$. By a direct computation, we can see that $\bar{c}_u \in P((P_3)_{2^{t+u}+2^u-3}^*)$ and $\langle \psi^u(\bar{p}_0), \bar{c}_u \rangle = 1$. So, we get

$$\mathbb{F}_2 \otimes_{GL_3} P((P_3)_n^*) = \begin{cases} \langle [\bar{c}_u] \rangle, & \text{if } s = 2, \\ 0, & \text{if } s \neq 2. \end{cases}$$

For $u = 0$, we have $\bar{c}_0 = a_1^{(2)} a_2^{(3)} a_3^{(3)} + a_1^{(1)} a_2^{(4)} a_3^{(3)} + a_1^{(1)} a_2^{(2)} a_3^{(5)} + a_1^{(1)} a_2^{(1)} a_3^{(6)}$.

A direct computation shows

$$\begin{aligned} \phi_3(a_1^{(2)} a_2^{(3)} a_3^{(3)}) &= \lambda_2 \lambda_3^2 + \lambda_1 \lambda_4 \lambda_3 + \lambda_1 \lambda_3 \lambda_4, \\ \phi_3(a_1^{(1)} a_2^{(4)} a_3^{(3)}) &= \lambda_1 \lambda_4 \lambda_3 + \lambda_1 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_5, \\ \phi_3(a_1^{(1)} a_2^{(2)} a_3^{(5)}) &= \lambda_1 \lambda_2 \lambda_5 + \lambda_1^2 \lambda_6, \\ \phi_3(a_1^{(1)} a_2^{(1)} a_3^{(6)}) &= \lambda_1^2 \lambda_6. \end{aligned}$$

Hence, we obtain $\phi_3(\bar{c}_0) = \lambda_2 \lambda_3^2$. By Theorem 2.2, we have $Tr_3([\bar{c}_0]) = [\lambda_2 \lambda_3^2] = c_0$.

Since $[\bar{c}_u] = (\widetilde{Sq}_*^0)^u([\bar{c}_0])$, we get

$$Tr_3([\bar{c}_u]) = Tr_3((\widetilde{Sq}_*^0)^u([\bar{c}_0])) = (Sq^0)^u Tr_3([\bar{c}_0]) = (Sq^0)^u(c_0) = c_u.$$

By Theorem 2.1, we have $h_u h_{u+1} = 0$. Hence,

$$\text{Ext}_A^{3, 2^{s+u+1}+2^{u+1}+2^u}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_u h_{u+1} h_{u+3}, c_u \rangle = \langle c_u \rangle, & \text{if } s = 2, \\ \langle h_u h_{u+1} h_{s+u+1} \rangle = 0, & \text{if } s \neq 2. \end{cases}$$

Theorem 3.2.1 in this case follows from the above equalities.

3.2.4. The case of the generic degree

In this subsection, we consider the degree

$$n = 2^{s+t+u} + 2^{t+u} + 2^u - 3,$$

with s, t, u non-negative integers.

The subcases either $s = 0$ or $t = 0$ have been determined in Subsections 3.2.1 and 3.2.2. The case $s > 0$ and $t = 1$ has been determined in Subsection 3.2.3. So, we assume that $s > 0$ and $t > 1$.

The iterated homomorphism

$$(\widetilde{Sq}_*^0)^u : (QP_3)_{2^{s+t+u}+2^{t+u}+2^u-3} \rightarrow (QP_3)_{2^{s+t}+2^t-2}$$

is an isomorphism of GL_3 -modules. So, we need only to compute $(QP_3)_{2^{s+t}+2^t-2}^{GL_3}$.

The subcase $s = 1$. Then $n = 2^{t+1} + 2^t - 2$. According to Kameko [23], $(QP_3)_n$ is the vector space with a basis consisting of all the classes represented by the following monomials:

$$\begin{aligned} v_{t,1} &= x_2^{2^t-1} x_3^{2^{t+1}-1} & v_{t,2} &= x_2^{2^{t+1}-1} x_3^{2^t-1} & v_{t,3} &= x_1^{2^t-1} x_3^{2^{t+1}-1} \\ v_{t,4} &= x_1^{2^t-1} x_2^{2^{t+1}-1} & v_{t,5} &= x_1^{2^{t+1}-1} x_3^{2^t-1} & v_{t,6} &= x_1^{2^{t+1}-1} x_2^{2^t-1} \\ v_{t,7} &= x_1 x_2^{2^t-2} x_3^{2^{t+1}-1} & v_{t,8} &= x_1 x_2^{2^{t+1}-1} x_3^{2^t-2} & v_{t,9} &= x_1^{2^{t+1}-1} x_2 x_3^{2^t-2} \\ v_{t,10} &= x_1 x_2^{2^t-1} x_3^{2^{t+1}-2} & v_{t,11} &= x_1 x_2^{2^{t+1}-2} x_3^{2^t-1} & v_{t,12} &= x_1^{2^t-1} x_2 x_3^{2^{t+1}-2} \\ v_{t,13} &= x_1^3 x_2^{2^{t+1}-3} x_3^{2^t-2}, \end{aligned}$$

$$v_{t,14} = x_1^3 x_2^2 x_3^4 \text{ for } t = 2, \text{ and } v_{t,14} = x_1^3 x_2^{2^t-3} x_3^{2^{t+1}-2} \text{ for } t > 2.$$

By a direct computation using the above basis, we obtain

Proposition 3.2.5 *For any integers $t > 1, u \geq 0$ and $n = 2^{t+u+1} + 2^{t+u} + 2^u - 3$, we have $(QP_3)_n^{GL_3} = 0$.*

By Theorem 2.1 $h_{t+u} h_{t+u+1} = 0$, so we have

$$\text{Ext}_A^{3,2^{t+u+1}+2^{t+u}+2^u}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_u h_{t+u} h_{t+u+1} \rangle = 0.$$

Hence, from the above equalities, we can see that

$$\text{Tr}_3 : \mathbb{F}_2 \otimes_{GL_3} P((P_3)_{2^{t+u+1}+2^{t+u}+2^u-3}^*) \longrightarrow \text{Ext}_A^{3,2^{t+u+1}+2^{t+u}+2^u}(\mathbb{F}_2, \mathbb{F}_2)$$

is a trivial isomorphism.

Now, suppose that $s, t > 1$ and $n = 2^{s+t} + 2^t - 2$. From the results of Kameko [23], we see that $(QP_3)_n$ is the vector space of dimension 21 with a basis consisting of all the classes represented by the following monomials:

$$\begin{aligned} v_{s,t,1} &= x_2^{2^t-1} x_3^{2^{s+t}-1} & v_{s,t,2} &= x_2^{2^{s+t}-1} x_3^{2^t-1} \\ v_{s,t,3} &= x_1^{2^t-1} x_3^{2^{s+t}-1} & v_{s,t,4} &= x_1^{2^t-1} x_2^{2^{s+t}-1} \\ v_{s,t,5} &= x_1^{2^{s+t}-1} x_3^{2^t-1} & v_{s,t,6} &= x_1^{2^{s+t}-1} x_2^{2^t-1} \\ v_{s,t,7} &= x_2^{2^{t+1}-1} x_3^{2^{s+t}-2^t-1} & v_{s,t,8} &= x_1^{2^{t+1}-1} x_3^{2^{s+t}-2^t-1} \\ v_{s,t,9} &= x_1^{2^{t+1}-1} x_2^{2^{s+t}-2^t-1} & v_{s,t,10} &= x_1 x_2^{2^t-2} x_3^{2^{s+t}-1} \\ v_{s,t,11} &= x_1 x_2^{2^{s+t}-1} x_3^{2^t-2} & v_{s,t,12} &= x_1^{2^{s+t}-1} x_2 x_3^{2^t-2} \\ v_{s,t,13} &= x_1 x_2^{2^t-1} x_3^{2^{s+t}-2} & v_{s,t,14} &= x_1 x_2^{2^{s+t}-2} x_3^{2^t-1} \\ v_{s,t,15} &= x_1^{2^t-1} x_2 x_3^{2^{s+t}-2} & v_{s,t,16} &= x_1 x_2^{2^{t+1}-2} x_3^{2^{s+t}-2^t-1} \\ v_{s,t,17} &= x_1 x_2^{2^{t+1}-1} x_3^{2^{s+t}-2^t-2} & v_{s,t,18} &= x_1^{2^{t+1}-1} x_2 x_3^{2^{s+t}-2^t-2} \\ v_{s,t,19} &= x_1^3 x_2^{2^{s+t}-3} x_3^{2^t-2} & v_{s,t,20} &= x_1^3 x_2^{2^{t+1}-3} x_3^{2^{s+t}-2^t-2}, \\ v_{s,2,21} &= x_1^3 x_2^2 x_3^{2^{s+2}-4}, \text{ for } t = 2 \text{ and } v_{s,t,21} = x_1^3 x_2^{2^t-3} x_3^{2^{s+t}-2} \text{ for } t > 2. \end{aligned}$$

We set

$$p_{3,s,t,u} = \begin{cases} \sum_{1 \leq j \leq 21, j \neq 13, 15} \psi^u(v_{s,2,j}), & \text{if } t = 2, \\ \sum_{1 \leq j \leq 21} \psi^u(v_{s,t,j}), & \text{if } t > 2. \end{cases}$$

By a direct computation using this basis, we get

Proposition 3.2.6 *For any integers $s, t > 1, u \geq 0$ and $n = 2^{s+t+u} + 2^{t+u} + 2^u - 3$, we have $(QP_3)_n^{GL_3} = [p_{3,s,t,u}]$.*

By Theorem 2.1, we have

$$\text{Ext}_A^{3,2^{s+t+u}+2^{t+u}+2^u}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_u h_{t+u} h_{s+t+u} \rangle.$$

Note that $\psi^u(v_{s,t,1}) = x_1^{2^u-1} x_2^{2^{t+u}-1} x_3^{2^{s+t+u}-1}$. Consider the element

$$q_{3,s,t,u} = a_1^{(2^u-1)} a_2^{(2^{t+u}-1)} a_3^{(2^{s+t+u}-1)} \in \mathbb{F}_2 \otimes_{GL_3} P((P_3)_n^*).$$

Since $\langle p_{3,s,t,u}, q_{3,s,t,u} \rangle = 1$, from Proposition 3.2.6, we obtain

$$\mathbb{F}_2 \otimes_{GL_3} P((P_3)_n^*) = \langle [q_{3,s,t,u}] \rangle.$$

It is easy to see that $\phi_3(q_{3,s,t,u}) = \lambda_u \lambda_{t+u} \lambda_{s+t+u}$, hence using Theorem 2.2 we get

$$Tr_3([q_{3,s,t,u}]) = [\lambda_u \lambda_{t+u} \lambda_{s+t+u}] = h_u h_{t+u} h_{s+t+u}.$$

Theorem 3.2.1 is completely proved.

4. Determination of Tr_4 in some internal degrees

In this section, we explicitly determined Tr_4 in some internal degrees. Our main result is the following.

Theorem 4.1 *Let s be a positive integer and let n be one of the degrees $2^{s+1} - 1, 2^{s+1} - 2, 2^{s+1} - 3$. Then, the homomorphism*

$$Tr_4 : \mathbb{F}_2 \otimes_{GL_4} P((P_4)_n^*) \longrightarrow \text{Ext}_{\mathcal{A}}^{4,n+4}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism.

We prove the theorem by computing the space $(QP_4)_n^{GL_4}$.

4.1. The case $n = 2^{s+1} - 3$

Proposition 4.1.1 (see [15, 26]) *Let $n = 2^{s+1} - 3$ with s a positive integer. Then, the dimension of the \mathbb{F}_2 -vector space $(QP_4)_n$ is determined by the following table:*

$n = 2^{s+1} - 3$	$s = 1$	$s = 2$	$s = 3$	$s \geq 4$
$\dim(QP_4)_n$	4	15	35	45

A basis for $(QP_4)_n$ is the set consisting of all the classes represented monomials $a_j = a_{s,j}$ which are determined as follows:

For $s = 1$, $a_{1,1} = x_4, a_{1,2} = x_3, a_{1,3} = x_2, a_{1,4} = x_1$.

For $s \geq 2$,

$$\begin{aligned} a_{s,1} &= x_2^{2^{s-1}-1} x_3^{2^{s-1}-1} x_4^{2^s-1} & a_{s,2} &= x_2^{2^{s-1}-1} x_3^{2^s-1} x_4^{2^{s-1}-1} \\ a_{s,3} &= x_2^{2^s-1} x_3^{2^{s-1}-1} x_4^{2^{s-1}-1} & a_{s,4} &= x_1^{2^{s-1}-1} x_3^{2^{s-1}-1} x_4^{2^s-1} \\ a_{s,5} &= x_1^{2^{s-1}-1} x_3^{2^s-1} x_4^{2^{s-1}-1} & a_{s,6} &= x_1^{2^{s-1}-1} x_2^{2^{s-1}-1} x_4^{2^s-1} \\ a_{s,7} &= x_1^{2^{s-1}-1} x_2^{2^{s-1}-1} x_3^{2^s-1} & a_{s,8} &= x_1^{2^{s-1}-1} x_2^{2^s-1} x_4^{2^{s-1}-1} \\ a_{s,9} &= x_1^{2^{s-1}-1} x_2^{2^s-1} x_3^{2^{s-1}-1} & a_{s,10} &= x_1^{2^s-1} x_3^{2^{s-1}-1} x_4^{2^{s-1}-1} \\ a_{s,11} &= x_1^{2^s-1} x_2^{2^{s-1}-1} x_4^{2^{s-1}-1} & a_{s,12} &= x_1^{2^s-1} x_2^{2^{s-1}-1} x_3^{2^{s-1}-1} \end{aligned}$$

For $s = 2$, $a_{2,13} = x_1 x_2 x_3 x_4^2, a_{2,14} = x_1 x_2 x_3^2 x_4, a_{2,15} = x_1 x_2^2 x_3 x_4$.

For $s \geq 3$,

$$\begin{aligned} a_{s,13} &= x_1 x_2^{2^{s-1}-2} x_3^{2^{s-1}-1} x_4^{2^s-1} & a_{s,14} &= x_1 x_2^{2^{s-1}-2} x_3^{2^s-1} x_4^{2^{s-1}-1} \\ a_{s,15} &= x_1 x_2^{2^{s-1}-1} x_3^{2^{s-1}-2} x_4^{2^s-1} & a_{s,16} &= x_1 x_2^{2^{s-1}-1} x_3^{2^s-1} x_4^{2^{s-1}-2} \\ a_{s,17} &= x_1 x_2^{2^s-1} x_3^{2^{s-1}-2} x_4^{2^{s-1}-1} & a_{s,18} &= x_1 x_2^{2^s-1} x_3^{2^{s-1}-1} x_4^{2^{s-1}-2} \\ a_{s,19} &= x_1^{2^{s-1}-1} x_2 x_3^{2^{s-1}-2} x_4^{2^s-1} & a_{s,20} &= x_1^{2^{s-1}-1} x_2 x_3^{2^s-1} x_4^{2^{s-1}-2} \\ a_{s,21} &= x_1^{2^{s-1}-1} x_2^{2^s-1} x_3 x_4^{2^{s-1}-2} & a_{s,22} &= x_1^{2^s-1} x_2 x_3^{2^{s-1}-2} x_4^{2^{s-1}-1} \\ a_{s,23} &= x_1^{2^s-1} x_2 x_3^{2^{s-1}-1} x_4^{2^{s-1}-2} & a_{s,24} &= x_1^{2^s-1} x_2^{2^{s-1}-1} x_3 x_4^{2^{s-1}-2} \\ a_{s,25} &= x_1 x_2^{2^{s-1}-1} x_3^{2^{s-1}-1} x_4^{2^s-2} & a_{s,26} &= x_1 x_2^{2^{s-1}-1} x_3^{2^s-2} x_4^{2^{s-1}-1} \\ a_{s,27} &= x_1 x_2^{2^s-2} x_3^{2^{s-1}-1} x_4^{2^{s-1}-1} & a_{s,28} &= x_1^{2^{s-1}-1} x_2 x_3^{2^{s-1}-1} x_4^{2^s-2} \\ a_{s,29} &= x_1^{2^{s-1}-1} x_2 x_3^{2^s-2} x_4^{2^{s-1}-1} & a_{s,30} &= x_1^{2^{s-1}-1} x_2^{2^{s-1}-1} x_3 x_4^{2^s-2} \end{aligned}$$

For $s = 3$,

$$\begin{aligned} a_{3,31} &= x_1^3 x_2^3 x_3^5 x_4^2 & a_{3,32} &= x_1^3 x_2^5 x_3^2 x_4^3 & a_{3,33} &= x_1^3 x_2^5 x_3^3 x_4^2 \\ a_{3,34} &= x_1^3 x_2^3 x_3^3 x_4^4 & a_{3,35} &= x_1^3 x_2^3 x_3^4 x_4^3 \end{aligned}$$

For $s \geq 4$,

$$\begin{aligned}
 a_{s,31} &= x_1^3 x_2^{2^{s-1}-3} x_3^{2^{s-1}-2} x_4^{2^s-1} & a_{s,32} &= x_1^3 x_2^{2^{s-1}-3} x_3^{2^s-1} x_4^{2^{s-1}-2} \\
 a_{s,33} &= x_1^3 x_2^{2^s-1} x_3^{2^{s-1}-3} x_4^{2^{s-1}-2} & a_{s,34} &= x_1^{2^s-1} x_2^3 x_3^{2^{s-1}-3} x_4^{2^{s-1}-2} \\
 a_{s,35} &= x_1^3 x_2^{2^{s-1}-3} x_3^{2^{s-1}-1} x_4^{2^s-2} & a_{s,36} &= x_1^3 x_2^{2^{s-1}-3} x_3^{2^s-2} x_4^{2^{s-1}-1} \\
 a_{s,37} &= x_1^3 x_2^{2^{s-1}-1} x_3^{2^{s-1}-3} x_4^{2^s-2} & a_{s,38} &= x_1^{2^{s-1}-1} x_2^3 x_3^{2^{s-1}-3} x_4^{2^s-2} \\
 a_{s,39} &= x_1^3 x_2^{2^{s-1}-1} x_3^{2^s-3} x_4^{2^{s-1}-2} & a_{s,40} &= x_1^3 x_2^{2^s-3} x_3^{2^{s-1}-2} x_4^{2^{s-1}-1} \\
 a_{s,41} &= x_1^3 x_2^{2^s-3} x_3^{2^{s-1}-1} x_4^{2^{s-1}-2} & a_{s,42} &= x_1^{2^{s-1}-1} x_2^3 x_3^{2^s-3} x_4^{2^{s-1}-2} \\
 a_{s,43} &= x_1^7 x_2^{2^s-5} x_3^{2^{s-1}-3} x_4^{2^{s-1}-2}
 \end{aligned}$$

For $s = 4$, $a_{4,44} = x_1^7 x_2^7 x_3^9 x_4^6$, $a_{4,45} = x_1^7 x_2^7 x_3^7 x_4^8$.

For $s \geq 5$, $a_{s,44} = x_1^7 x_2^{2^{s-1}-5} x_3^{2^s-3} x_4^{2^{s-1}-2}$, $a_{s,45} = x_1^7 x_2^{2^{s-1}-5} x_3^{2^{s-1}-3} x_4^{2^s-2}$.

Proposition 4.1.2 *Let s be a positive integer. Then, $(QP_4)_{2^{s+1}-3}^{GL_4} = 0$.*

For simplicity, we prove the proposition in detail for $s \geq 5$. The other cases can be proved by the similar computations.

For any monomials z_1, z_2, \dots, z_m in P_k and for a subgroup $G \subset GL_k$, we denote $G(z_1, z_2, \dots, z_m)$ the G -submodule of QP_k generated by the set $\{[z_i] : 1 \leq i \leq m\}$. We have the following.

Lemma 4.1.3 i) *For any $s \geq 5$, there is a direct summand decomposition of the Σ_4 -modules:*

$$(QP_4)_{2^{s+1}-3} = \Sigma_4(a_{s,1}) \oplus \Sigma_4(a_{s,13}) \oplus \Sigma_4(a_{s,31}) \oplus \Sigma_4(a_{s,25}, a_{s,35}, a_{s,43}).$$

(ii) $\Sigma_4(a_{s,1})^{\Sigma_4} = \langle [p_{4,s,1}] \rangle$, with $p_{4,s,1} = \sum_{j=1}^{12} a_{s,j}$.

(iii) $\Sigma_4(a_{s,13})^{\Sigma_4} = \langle [p_{4,s,2}] \rangle$, with $p_{4,s,2} = \sum_{j=13}^{24} a_{s,j}$.

(iv) $\Sigma_4(a_{s,31})^{\Sigma_4} = \langle [p_{4,s,3}] \rangle$, with $p_{4,s,3} = \sum_{j=31}^{34} a_{s,j}$.

(v) $\Sigma_4(a_{s,25}, a_{s,35}, a_{s,43})^{\Sigma_4} = \langle [p_{4,s,4}] \rangle$, with $p_{4,s,4} = \sum_{j=25}^{30} a_{s,j} + \sum_{j=39}^{43} a_{s,j} + a_{s,45}$.

Proof. We obtain Part i) by a simple computation using Proposition 4.1.1. We prove Part v) in detail. The others can be proved by the similar computations. By a simple computation we see that the set $\{[a_{s,j}] : j = 25, \dots, 30, 35, \dots, 45\}$ is a basis for $\Sigma_4(a_{s,25}, a_{s,35}, a_{s,43})$. Suppose $[f] \in \Sigma_4(a_{s,25}, a_{s,35}, a_{s,43})^{\Sigma_4}$, then

$$f \equiv \sum_{j=25}^{30} \gamma_j a_{s,j} + \sum_{j=35}^{45} \gamma_j a_{s,j}$$

with $\gamma_j \in \mathbb{F}_2$. By a direct computation, we get

$$\begin{aligned}
 \rho_1(f) + f &\equiv (\gamma_{25} + \gamma_{28})(a_{s,25} + a_{s,28}) + (\gamma_{26} + \gamma_{29})(a_{s,26} + a_{s,29}) \\
 &\quad + (\gamma_{27} + \gamma_{41})a_{s,35} + (\gamma_{27} + \gamma_{40})a_{s,36} + (\gamma_{37} + \gamma_{38})(a_{s,37} + a_{s,38}) \\
 &\quad + (\gamma_{39} + \gamma_{42})(a_{s,39} + a_{s,42}) + (\gamma_{41} + \gamma_{43})a_{s,44} + (\gamma_{40} + \gamma_{43})a_{s,45} \equiv 0,
 \end{aligned}$$

$$\begin{aligned}
 \rho_2(f) + f &\equiv (\gamma_{26} + \gamma_{27})(a_{s,26} + a_{s,27}) + (\gamma_{28} + \gamma_{30})(a_{s,28} + a_{s,30}) \\
 &\quad + (\gamma_{35} + \gamma_{37})(a_{s,35} + a_{s,37}) + (\gamma_{29} + \gamma_{36} + \gamma_{40})(a_{s,36} + a_{s,40}) \\
 &\quad + (\gamma_{39} + \gamma_{41})(a_{s,39} + a_{s,41}) + (\gamma_{42} + \gamma_{43} + \gamma_{44})(a_{s,43} + a_{s,44}) \\
 &\quad + (\gamma_{29} + \gamma_{42})(a_{s,38} + a_{s,45}) \equiv 0,
 \end{aligned}$$

$$\begin{aligned}
 \rho_3(f) + f &\equiv (\gamma_{25} + \gamma_{26})(a_{s,25} + a_{s,26}) + (\gamma_{28} + \gamma_{29})(a_{s,28} + a_{s,29}) \\
 &\quad + (\gamma_{35} + \gamma_{36})(a_{s,35} + a_{s,36}) + (\gamma_{30} + \gamma_{37} + \gamma_{39})(a_{s,37} + a_{s,39}) \\
 &\quad + (\gamma_{30} + \gamma_{38} + \gamma_{42})(a_{s,38} + a_{s,42}) + (\gamma_{40} + \gamma_{41})(a_{s,40} + a_{s,41}) \\
 &\quad + (\gamma_{30} + \gamma_{44} + \gamma_{45})(a_{s,44} + a_{s,45}) \equiv 0.
 \end{aligned}$$

The above equalities imply $\gamma_j = 0$ for $j = 35, 36, 37, 38, 44$ and $\gamma_j = \gamma_{25}$ for $j \neq 35, 36, 37, 38, 44$. The lemma is proved. □

Proof of Proposition 4.1.2. Let $f \in P_4$ such that $[f] \in (QP_4)^{GL_4}_{2^{s+1}-3}$. Since $\Sigma_4 \subset GL_4$, we have $[f] \in (QP_4)^{\Sigma_4}_{2^{s+1}-3}$. Then, $f \equiv \sum_{j=1}^4 \gamma_j p_{4,s,j}$ with $\gamma_j \in \mathbb{F}_2$. By a direct computation, we get

$$\begin{aligned} \rho_4(f) + f &\equiv (\gamma_1 + \gamma_4)a_{s,3} + \gamma_1 a_{s,9} + (\gamma_2 + \gamma_3)a_{s,15} \\ &\quad + \gamma_2 a_{s,21} + \text{other terms} \equiv 0. \end{aligned}$$

The last equality implies $\gamma_j = 0$ for $j = 1, 2, 3, 4$. The proposition follows. □

From Theorem 2.1, we have $\text{Ext}_{\mathcal{A}}^{4,2^{s+1}+1}(\mathbb{F}_2, \mathbb{F}_2) = 0$. Hence, Theorem 4.1 holds for $n = 2^{s+1} - 3$.

4.2. The case $n = 2^{s+1} - 2$

Since Kameko's homomorphism in the degree $2^{s+1} - 2$,

$$(\widetilde{Sq}_*^0)_{2^{s+1}-2} : (QP_4)_{2^{s+1}-2} \rightarrow (QP_4)_{2^s-3}$$

is an epimorphism of GL_4 -modules, using Proposition 4.1.2, we have

$$(QP_4)^{GL_4}_{2^{s+1}-2} \subset (\text{Ker}(\widetilde{Sq}_*^0)_{2^{s+1}-2})^{GL_4}.$$

From [15, 26], we have the following.

Proposition 4.2.1 (see [15, 26]) *Let s be a positive integer. Then,*

$$\dim(\text{Ker}(\widetilde{Sq}_*^0)_{2^{s+1}-2}) = \begin{cases} 6, & \text{if } s = 1, \\ 20, & \text{if } s = 2, \\ 35 & \text{if } s \geq 3. \end{cases}$$

A basis for $(\text{Ker}(\widetilde{Sq}_*^0)_{2^{s+1}-2})$ is the set consisting of all the classes represented monomials $b_j = b_{s,j}$ which are determined as follows:

For $s \geq 1$,

$$\begin{aligned} b_{s,1} &= x_3^{2^s-1} x_4^{2^s-1} & b_{s,2} &= x_2^{2^s-1} x_4^{2^s-1} & b_{s,3} &= x_2^{2^s-1} x_3^{2^s-1} \\ b_{s,4} &= x_1^{2^s-1} x_4^{2^s-1} & b_{s,5} &= x_1^{2^s-1} x_3^{2^s-1} & b_{s,6} &= x_1^{2^s-1} x_2^{2^s-1} \end{aligned}$$

For $s \geq 2$,

$$\begin{aligned} b_{s,7} &= x_2 x_3^{2^s-2} x_4^{2^s-1} & b_{s,8} &= x_2 x_3^{2^s-1} x_4^{2^s-2} & b_{s,9} &= x_2^{2^s-1} x_3 x_4^{2^s-2} \\ b_{s,10} &= x_1 x_3^{2^s-2} x_4^{2^s-1} & b_{s,11} &= x_1 x_3^{2^s-1} x_4^{2^s-2} & b_{s,12} &= x_1 x_2^{2^s-2} x_4^{2^s-1} \\ b_{s,13} &= x_1 x_2^{2^s-2} x_3^{2^s-1} & b_{s,14} &= x_1 x_2^{2^s-1} x_4^{2^s-2} & b_{s,15} &= x_1 x_2^{2^s-1} x_3^{2^s-2} \\ b_{s,16} &= x_1^{2^s-1} x_3 x_4^{2^s-2} & b_{s,17} &= x_1^{2^s-1} x_2 x_4^{2^s-2} & b_{s,18} &= x_1^{2^s-1} x_2 x_3^{2^s-2} \end{aligned}$$

For $s = 2$, $b_{2,19} = x_1 x_2 x_3^2 x_4^2$, $b_{2,20} = x_1 x_2^2 x_3 x_4^2$.

For $s \geq 3$,

$$\begin{aligned} b_{s,19} &= x_2^3 x_3^{2^s-3} x_4^{2^s-2} & b_{s,20} &= x_1^3 x_3^{2^s-3} x_4^{2^s-2} & b_{s,21} &= x_1^3 x_2^{2^s-3} x_4^{2^s-2} \\ b_{s,22} &= x_1^3 x_2^{2^s-3} x_3^{2^s-2} & b_{s,23} &= x_1 x_2^2 x_3^{2^s-4} x_4^{2^s-1} & b_{s,24} &= x_1 x_2^2 x_3^{2^s-1} x_4^{2^s-4} \\ b_{s,25} &= x_1 x_2^{2^s-1} x_3^2 x_4^{2^s-4} & b_{s,26} &= x_1^{2^s-1} x_2 x_3^2 x_4^{2^s-4} & b_{s,27} &= x_1 x_2 x_3^{2^s-2} x_4^{2^s-2} \\ b_{s,28} &= x_1 x_2^{2^s-2} x_3 x_4^{2^s-2} & b_{s,29} &= x_1^3 x_2^5 x_3^{2^s-6} x_4^{2^s-4} & b_{s,30} &= x_1 x_2^2 x_3^{2^s-3} x_4^{2^s-2} \\ b_{s,31} &= x_1 x_2^3 x_3^{2^s-4} x_4^{2^s-2} & b_{s,32} &= x_1 x_2^3 x_3^{2^s-2} x_4^{2^s-4} & b_{s,33} &= x_1^3 x_2 x_3^{2^s-4} x_4^{2^s-2} \\ b_{s,34} &= x_1^3 x_2 x_3^{2^s-2} x_4^{2^s-4} \end{aligned}$$

For $s = 3$, $b_{3,35} = x_1^3 x_2^3 x_3^4 x_4^4$, and for $s \geq 4$, $b_{s,35} = x_1^3 x_2^{2^s-3} x_3^2 x_4^{2^s-4}$.

We set

$$p_{4,s} = \begin{cases} x_1 x_2 x_3^6 x_4^6 + x_1^3 x_2^3 x_3^4 x_4^4, & \text{if } s = 3, \\ \sum_{j=1}^{35} b_{s,j} & \text{if } s \geq 4. \end{cases}$$

By a direct computation using Proposition 4.2.1, one gets the following.

Proposition 4.2.2 *Let s be a positive integer. Then,*

$$(\text{Ker}(\widetilde{Sq}_*^0)_{2^{s+1}-2})^{GL_4} = \begin{cases} 0, & \text{if } s \leq 2, \\ \langle [p_{4,s}] \rangle & \text{if } s \geq 3. \end{cases}$$

For simplicity, we will prove this proposition in detail for $s \geq 4$. The others can be proved by the similar computations. We have the following.

Lemma 4.2.3 i) For any $s \geq 4$, there is a direct summand decomposition of the Σ_4 -modules:

$$(\text{Ker}(\widetilde{Sq}_*)_{2^{s+1}-2} = \Sigma_4(b_{s,1}) \oplus \Sigma_4(b_{s,7}) \oplus \Sigma_4(b_{s,19}) \oplus \Sigma_4(b_{s,23}) \oplus \Sigma_4(b_{s,29}, b_{s,30}).$$

$$(ii) \Sigma_4(b_{s,1})^{\Sigma_4} = \langle [\bar{p}_{4,s,1}] \rangle, \text{ with } \bar{p}_{4,s,1} = \sum_{j=1}^6 b_{s,j}.$$

$$(iii) \Sigma_4(b_{s,7})^{\Sigma_4} = \langle [\bar{p}_{4,s,2}] \rangle, \text{ with } \bar{p}_{4,s,2} = \sum_{j=7}^{18} b_{s,j}.$$

$$(iv) \Sigma_4(b_{s,19})^{\Sigma_4} = \langle [\bar{p}_{4,s,3}] \rangle, \text{ with } \bar{p}_{4,s,3} = \sum_{j=19}^{22} b_{s,j}.$$

$$(v) \Sigma_4(b_{s,23})^{\Sigma_4} = \langle [\bar{p}_{4,s,4}] \rangle, \text{ with } \bar{p}_{4,s,4} = \sum_{j=23}^{26} b_{s,j}.$$

$$(vi) \Sigma_4(b_{s,29}, a_{s,30})^{\Sigma_4} = \langle [\bar{p}_{4,s,5}], [\bar{p}_{4,s,6}] \rangle, \text{ where}$$

$$\bar{p}_{4,s,5} = \sum_{j=27}^{29} b_{s,j}, \quad \bar{p}_{4,s,6} = \sum_{j=30}^{35} b_{s,j}.$$

Proof. From Proposition 4.2.1 we easily obtain Part i). Now, we prove Part vi) in detail. The others can be proved by the similar computations. By a direct computation we see that the set $\{[b_{s,j}] : j = 27 \leq j \leq 35\}$ is a basis for $\Sigma_4(b_{s,29}, b_{s,30})^{\Sigma_4}$. Suppose $[f] \in \Sigma_4(b_{s,29}, b_{s,30})^{\Sigma_4}$, then $f \equiv \sum_{j=27}^{35} \gamma_j b_{s,j}$ with $\gamma_j \in \mathbb{F}_2$. By a direct computation, we obtain

$$\begin{aligned} \rho_1(f) + f &\equiv (\gamma_{28} + \gamma_{29} + \gamma_{30} + \gamma_{35})b_{s,27} + (\gamma_{31} + \gamma_{33})(b_{s,31} + b_{s,33}) \\ &\quad + (\gamma_{32} + \gamma_{34})(b_{s,32} + b_{s,34}) \equiv 0, \end{aligned}$$

$$\begin{aligned} \rho_2(f) + f &\equiv (\gamma_{27} + \gamma_{28} + \gamma_{32} + \gamma_{33})(b_{s,27} + b_{s,28}) \\ &\quad + (\gamma_{30} + \gamma_{31})(b_{s,30} + b_{s,31}) + (\gamma_{34} + \gamma_{35})(b_{s,34} + b_{s,35}) \equiv 0, \end{aligned}$$

$$\begin{aligned} \rho_3(f) + f &\equiv (\gamma_{28} + \gamma_{29} + \gamma_{30} + \gamma_{35})b_{s,27} + (\gamma_{31} + \gamma_{32})(b_{s,31} + b_{s,32}) \\ &\quad + (\gamma_{33} + \gamma_{34})(b_{s,33} + b_{s,34}) \equiv 0. \end{aligned}$$

The above equalities imply $\gamma_j = \gamma_{27}$ for $j = 27, 28, 29$ and $\gamma_j = \gamma_{30}$ for $30 \leq j \leq 35$. The lemma is proved. □

Remark 4.2.4 For $s = 3$, Parts i) to v) of Lemma 4.2.3 hold. We replace Part vi) with $\Sigma_4(b_{3,29}, a_{3,30})^{\Sigma_4} = [p_{4,3}]$.

Proof of Proposition 4.2.2. Let $f \in P_4$ such that $[f] \in \text{Ker}(\widetilde{Sq}_*)_{2^{s+1}-2}^{GL_4}$. Then, $[f] \in \text{Ker}(\widetilde{Sq}_*)_{2^{s+1}-2}^{\Sigma_4}$. Hence, $f \equiv \sum_{j=1}^6 \gamma_j \bar{p}_{4,s,j}$ with $\gamma_j \in \mathbb{F}_2$. By a direct computation, we have

$$\begin{aligned} \rho_4(f) + f &\equiv (\gamma_1 + \gamma_2)(b_{s,2} + b_{s,3}) + (\gamma_2 + \gamma_4)(b_{s,7} + b_{s,8}) + (\gamma_2 + \gamma_5)b_{s,9} \\ &\quad + (\gamma_2 + \gamma_3)(b_{s,14} + b_{s,15}) + (\gamma_3 + \gamma_6)b_{s,19} + (\gamma_4 + \gamma_6)b_{s,25} \\ &\quad + (\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5)b_{s,27} + (\gamma_5 + \gamma_6)(b_{s,31} + b_{s,32}) \equiv 0. \end{aligned}$$

The last equality implies $\gamma_j = \gamma_1$ for $1 \leq j \leq 6$. The proposition follows. □

From Theorem 2.1, we have

$$\text{Ext}_{\mathcal{A}}^{4,2^{s+1}+2}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 0, & \text{if } s \leq 2, \\ \langle d_0 \rangle & \text{if } s = 3, \\ \langle h_0^2 h_s^2 \rangle & \text{if } s \geq 4. \end{cases}$$

Denote $q_{4,s} \in P((P_4^*)_{2^{s+1}-2})$ by setting

$$\begin{aligned} q_{4,3} &= a_1^{(1)} a_2^{(1)} a_3^{(6)} a_4^{(6)} + a_1^{(1)} a_2^{(2)} a_3^{(5)} a_4^{(6)} + a_1^{(1)} a_2^{(3)} a_3^{(4)} a_4^{(6)} + a_1^{(1)} a_2^{(4)} a_3^{(3)} a_4^{(6)} + a_1^{(1)} a_2^{(5)} a_3^{(2)} a_4^{(6)} + a_1^{(1)} a_2^{(6)} a_3^{(1)} a_4^{(6)} \\ &\quad + a_1^{(2)} a_2^{(1)} a_3^{(6)} a_4^{(5)} + a_1^{(2)} a_2^{(2)} a_3^{(5)} a_4^{(5)} + a_1^{(2)} a_2^{(3)} a_3^{(4)} a_4^{(5)} + a_1^{(2)} a_2^{(4)} a_3^{(3)} a_4^{(5)} + a_1^{(2)} a_2^{(5)} a_3^{(2)} a_4^{(5)} + a_1^{(2)} a_2^{(6)} a_3^{(1)} a_4^{(5)} \\ &\quad + a_1^{(3)} a_2^{(1)} a_3^{(5)} a_4^{(5)} + a_1^{(3)} a_2^{(2)} a_3^{(6)} a_4^{(3)} + a_1^{(3)} a_2^{(3)} a_3^{(2)} a_4^{(6)} + a_1^{(3)} a_2^{(4)} a_3^{(1)} a_4^{(6)} + a_1^{(3)} a_2^{(4)} a_3^{(2)} a_4^{(5)} a_1^{(3)} a_2^{(4)} a_3^{(4)} a_4^{(3)} \\ &\quad + a_1^{(3)} a_2^{(6)} a_3^{(2)} a_4^{(3)} + a_1^{(4)} a_2^{(1)} a_3^{(6)} a_4^{(3)} + a_1^{(4)} a_2^{(2)} a_3^{(5)} a_4^{(3)} + a_1^{(4)} a_2^{(3)} a_3^{(4)} a_4^{(3)} + a_1^{(4)} a_2^{(4)} a_3^{(3)} a_4^{(3)} + a_1^{(4)} a_2^{(5)} a_3^{(2)} a_4^{(3)} \end{aligned}$$

$$\begin{aligned}
 &+ a_1^{(4)} a_2^{(6)} a_3^{(1)} a_4^{(3)} + a_1^{(5)} a_2^{(1)} a_3^{(3)} a_4^{(5)} + a_1^{(5)} a_2^{(2)} a_3^{(1)} a_4^{(6)} + a_1^{(5)} a_2^{(2)} a_3^{(2)} a_4^{(5)} + a_1^{(5)} a_2^{(2)} a_3^{(4)} a_4^{(3)} + a_1^{(5)} a_2^{(3)} a_3^{(1)} a_4^{(5)} \\
 &+ a_1^{(5)} a_2^{(3)} a_3^{(3)} a_4^{(3)} + a_1^{(5)} a_2^{(5)} a_3^{(1)} a_4^{(3)} + a_1^{(6)} a_2^{(1)} a_3^{(1)} a_4^{(6)} + a_1^{(6)} a_2^{(1)} a_3^{(2)} a_4^{(5)} + a_1^{(6)} a_2^{(1)} a_3^{(4)} a_4^{(3)} + a_1^{(6)} a_2^{(2)} a_3^{(3)} a_4^{(3)},
 \end{aligned}$$

and $q_{4,s} = a_1^{(0)} a_2^{(0)} a_3^{(2^s-1)} a_4^{(2^s-1)}$, for $s \geq 4$. Since $\langle [p_{4,s}], [q_{4,s}] \rangle = 1$ for any $s \geq 3$, we get

$$\mathbb{F}_2 \otimes_{GL_4} P((P_4)_{2^{s+1}-2})^* = \begin{cases} 0, & \text{if } s \leq 2, \\ \langle [q_{4,s}] \rangle, & \text{if } s \geq 3. \end{cases}$$

By a direct computation, we obtain

$$\phi_4(q_{4,s}) = \begin{cases} \bar{d}_0 + \delta(\lambda_1 \lambda_9 \lambda_3^2 + \lambda_1 \lambda_3 \lambda_9 \lambda_3), & \text{if } s = 3, \\ \lambda_0^2 \lambda_{2^s-1}^2, & \text{if } s > 3. \end{cases}$$

From the above equalities and Theorem 2.2, we get

$$Tr_4([q_{4,s}]) = [\phi_3(q_{4,s})] = \begin{cases} [\bar{d}_0] = d_0, & \text{if } s = 3, \\ [\lambda_0^2 \lambda_{2^s-1}^2] = h_0^2 h_s^2, & \text{if } s > 3. \end{cases}$$

Theorem 4.1 holds for $n = 2^{s+1} - 2$.

4.3. The case $n = 2^{s+1} - 1$

First, we recall the following.

Proposition 4.3.1 (see [15, 26]) *Let $n = 2^{s+1} - 1$ with s a positive integer. Then, the dimension of the \mathbb{F}_2 -vector space $(QP_4)_n$ is determined by the following table:*

$n = 2^s - 1$	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s \geq 5$
$\dim(QP_4)_n$	14	35	75	89	85

A basis of $(QP_4)_n$ has been given in [15]. For $s \geq k - 2$, we set

$$\eta_{k,s} = \sum_{m=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_m \leq k} x_{i_1} x_{i_2}^2 \dots x_{i_{m-1}}^{2^{m-2}} x_{i_m}^{2^{s+1}-2^{m-1}} \in (P_k)_{2^{s+1}-1}.$$

For $k = 4$, we denote

$$\bar{p}_{4,s} = \begin{cases} \eta_{4,s} + x_1 x_2^2 x_3^2 x_4^2, & \text{if } s = 2, \\ \eta_{4,s} + x_1 x_2^2 x_3^4 x_4^{2^{s+1}-8}, & \text{if } s \geq 3. \end{cases}$$

By a computation similar to the one in Proposition 4.2.1, one gets the following.

Proposition 4.3.2 *Let s be a positive integer. Then,*

$$(QP_4)_{2^{s+1}-1}^{GL_4} = \begin{cases} 0, & \text{if } s = 1, \\ \langle [\bar{p}_{4,s}] \rangle, & \text{if } s \geq 2. \end{cases}$$

From Theorem 2.1, we have

$$Ext_A^{4, 2^{s+1}+3}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 0, & \text{if } s = 1, \\ \langle h_0^3 h_{s+1} \rangle, & \text{if } s \geq 2. \end{cases}$$

Denote $\bar{q}_{4,s} = a_1^{(0)} a_2^{(0)} a_3^{(0)} a_4^{(2^{s+1}-1)} \in P((P_4)_{2^{s+1}-1})^*$, for $s \geq 2$. It is easy to see that $\langle [\bar{p}_{4,s}], [\bar{q}_{4,s}] \rangle = 1$. Hence, we obtain

$$\mathbb{F}_2 \otimes_{GL_4} P((P_4)_{2^{s+1}-1})^* = \begin{cases} 0, & \text{if } s = 1, \\ \langle [\bar{q}_{4,s}] \rangle, & \text{if } s \geq 2. \end{cases}$$

By a simple computation, we have $\phi_4(\bar{q}_{4,s}) = \lambda_0^3 \lambda_{2^{s+1}-1}$. Hence, using Theorem 2.2, one gets

$$Tr_4([\bar{q}_{4,s}]) = [\phi_4(\bar{q}_{4,s})] = [\lambda_0^3 \lambda_{2^{s+1}-1}] = h_0^3 h_{s+1}.$$

Theorem 4.1 is completely proved.

ACKNOWLEDGEMENTS

This article was written when the author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) from August to November 2017. He would like to thank the VIASM for supporting the visit, convenient working condition and for kind hospitality.

The author would like to express his warmest thanks to the referee for carefully reading the manuscript and giving many criticisms and suggestions, which have led to an improvement of the article's exposition.

REFERENCES

- [1] W.M. Singer (1989), "The transfer in homological algebra", *Math. Zeit.*, **202(4)**, pp.493-523, MR1022818.
- [2] J.M. Boardman (1993), "Modular representations on the homology of powers of real projective spaces", *Contemp. Math.*, **146**, pp.49-70, MR1224907.
- [3] R.R. Bruner, L.M. Ha, N.H.V. Hung (2005), "On the behavior of the algebraic transfer", *Trans. of the Amer. Math. Soc.*, **357(2)**, pp.473-487, MR2095619.
- [4] P.H. Chon, L.M. Ha (2014), "On the May spectral sequence and the algebraic transfer II", *Topology Appl.*, **178**, pp.372-383, MR3276753.
- [5] L.M. Ha (2007), "Sub-Hopf algebras of the Steenrod algebra and the Singer transfer", *Geom. Topol. Monogr.*, **11**, pp.81-105, MR2402802.
- [6] N.H.V. Hung, V.T.N. Quynh (2009), "The image of Singer's fourth transfer", *Comptes Rendus Mathematique*, **347(23-24)**, pp.1415-1418, MR2588792.
- [7] T.N. Nam (2008), "Transfert algébrique et action d'ugroupe linéaire sur les puissances divisées modulo 2", *Annales de l'institut Fourier*, **58(5)**, pp.1785-1837, MR2445834.
- [8] N.E. Steenrod, D.B.A. Epstein (1962), *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J., MR0145525.
- [9] N.H.V. Hung (2005), "The cohomology of the Steenrod algebra and representations of the general linear groups", *Trans. of the Amer. Math. Soc.*, **357(10)**, pp.4065-4089, MR2159700.
- [10] P.H. Chon, L.M. Ha (2011), "Lambda algebra and the Singer transfer", *Comptes Rendus Mathematique*, **349(21-23)**, pp.21-23, MR2755689.
- [11] P.H. Chon, L.M. Ha (2012), "On May spectral sequence and the algebraic transfer", *Manuscripta Math.*, **138(1-2)**, pp.141-160, MR2898751.
- [12] N. Minami (1999), "The iterated transfer analogue of the new doomsday conjecture", *Trans. of the Amer. Math. Soc.*, **351(6)**, pp.2325-2351, MR1443884.
- [13] N. Sum (2014), "On the Peterson hit problem of five variables and its applications to the fifth Singer transfer", *East-West J. of Mathematics*, **16(1)**, pp.47-62, MR3409252.
- [14] N. Sum (2010), "The negative answer to Kameko's conjecture on the hit problem", *Adv. in Math.*, **225**, pp.2365-2390, MR2680169.
- [15] N. Sum (2015), "On the Peterson hit problem", *Adv. in Math.*, **274**, pp.432-489, MR3318156.
- [16] A.K. Bousfield, et al. (1966), "The mod- p lower central series and the Adams spectral sequence", *Topology*, **5(4)**, pp.331-342, MR0199862.
- [17] J. Adem (1952), "The iteration of the Steenrod squares in algebraic topology", *Proc. Nat. Acad. Sci. U.S.A.*, **38(8)**, pp.720-726, MR0050278.
- [18] J.F. Adams (1960), "On the non-existence of elements of Hopf invariant one", *Ann. of Math.*, **72(1)**, pp.20-104, MR0141119.
- [19] C.T.C. Wall (1960), "Generators and relations for the Steenrod algebra", *Ann. of Math.*, **72(3)**, pp.429-444, MR0116326.
- [20] J.S.P. Wang (1967), "On the cohomology of the mod-2 Steenrod algebra and the non-existence of mappings of Hopf invariant one", *Illinois Journal of Mathematics*, **11(3)**, pp.480-490, MR0214065.
- [21] W.H. Lin (2008), " $\text{Ext}_{\mathcal{A}}^{4,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ and $\text{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ ", *Topology Appl.*, **155(5)**, pp.459-496, MR2380930.
- [22] N.H.V. Hung (1999), "The weak conjecture on spherical classes", *Math. Zeit.*, **231(4)**, pp.727-743, MR1709493.
- [23] M. Kameko (1990), *Products of projective spaces as Steenrod modules*, PhD. Thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 29 pp, MR2638633.
- [24] R.M.W. Wood (1989), "Steenrod squares of polynomials and the Peterson conjecture", *Math. Proc. of the Cambriges Phil. Soc.*, **105(2)**, pp.307-309, MR0974986.
- [25] F.P. Peterson (1987), "Generators of $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ as a module over the Steenrod algebra", *Abstracts Amer. Math. Soc.*, **833**, pp.55-89.
- [26] N. Sum (2007), *The hit problem for the polynomial algebra of four variables*, available online at: <http://arxiv.org/abs/1412.1709>, 240 pp.