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Finite-difference method for the Gamma equation on non-uniform grids

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Abstract:

We propose a new monotone finite-difference scheme for the second-order local approximation on a non-uniform grid that approximates the Dirichlet initial boundary value problem (IBVP) for the quasi-linear convection-diffusion equation with unbounded nonlinearity, namely, for the Gamma equation obtained by transformation of the nonlinear Black-Scholes equation into a quasilinear parabolic equation. Using the difference maximum principle, a two-sided estimate and an a priori estimate in the *C*-norm are obtained for the solution of the difference schemes that approximate this equation.

Keywords: **Gamma equation, maximum principle, monotone finite-difference scheme, non-uniform grid, quasi-linear parabolic equation, scientific computing, two-side estimates.**

Classification number: **1.1**

Introduction

In the theory of difference schemes [1-3], the maximum principle is used to study the stability and convergence of a difference solution in the uniform norm. Computational methods that satisfy the maximum principle are usually called monotone [1, 2]. The monotone schemes play an critical role in computational practice. They make it possible to obtain a numerical solution without oscillations even in the case of non-smooth solutions [4].

When constructing monotone difference schemes, it is desirable to preserve the second order approximation with respect to the spatial variable. Such schemes are constructed for parabolic and hyperbolic equations in the presence of lower derivatives. For example, a nonconservative scheme of second order approximation for linear parabolic equations of general form on uniform grids is given in [1, 2]. When solving two-dimensional partial differential equations in the free domain, we need to construct a difference scheme on a non-uniform grid. We must first confirm that a non-uniform grid is more general than a uniform grid. While one can easily convert a non-uniform grid to uniform grid, the inverse transformation is not so straightforward, and it cannot preserve the conservation properties [5]. For the nonlinear Black-Scholes equation, it is helpful to implement the grid to the payoff of the option, because the price of an option may be more sensitive in a precise area [6]. In this case, the uniform grid is not appropriate. In the case of non-uniform grids for equations in mathematical physics with variable coefficients without lower derivatives, a scheme was obtained in [7] for which the conditions of the maximum principle are fulfilled without relations on the coefficients and parameters of the grid (unconditional monotonicity). In [8], the unconditionally monotone and economical schemes of second order approximation were constructed on a non-uniform grid for non-stationary multidimensional convection-diffusion problems.

In the present work, the previously obtained results are

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generalized to the construction of monotone difference schemes of second-order local approximation on non-uniform spatial grids for the Gamma equation for the second derivative of the option price in financial mathematics [9, 10]. The construction of such schemes is based on the appropriate choice of the perturbed coefficient, similar to [1, 2, 8]. Using the difference maximum principle, two-sided and a priori estimates are obtained in the normal C for solving difference schemes that approximate the above equation.

Problem setting and two-sided estimate of the exact solution

We consider the following quasilinear parabolic equation, which is called the Gamma equation [9, 10]:

$$\frac{\partial u}{\partial t}=\frac{\partial^2\beta(u)}{\partial x^2}+\frac{\partial\beta(u)}{\partial x}+c\frac{\partial u}{\partial x},\quad u=u(x,t),\quad 0<t\leqslant T,\quad x\in\mathbb{R},\quad c-const, \quad (1)$$

$$u(-\infty,t)=u(+\infty,t)=0,\quad u(x,0)=u_0(x). \quad (2)$$

According to the studies in [9, 10], Eqs. (1)-(2) are obtained by transforming the nonlinear Black-Scholes equation for V(S,τ) such that

$$V_\tau+0.5\sigma^2(t,S,V_{SS})S^2V_{SS}+(c-q)SV_{SS}-cV=0,\quad 0\leqslant S<\infty,\quad 0\leqslant\tau\leqslant T. \quad (3)$$

The present paper will focus on some models related to nonlinear Black-Scholes equations for the European option whose volatility relies upon various factors like the stock price, the option price, the time, as well as their derivatives, due to the presence of transaction cost. The option's behaviour would be disclosed by a higher derivative of its price, which is mentioned as the Greeks in the financial literature. Reliable numerical methods are not only useful for providing a good approximation for the pricing option, but they are also essential for its derivatives because of the relevance of the Greeks to quantitative analysis.

For the case of European call options [10], V(S,τ) is a solution of Eq. (3) with q=0 and 0⩽S<∞, 0⩽τ⩽T. The initial condition and boundary conditions of the problem in Eq. (3) will be

$$V(S,T)=\max\{0,S-E\},\quad 0\leqslant S<\infty,\quad E>0,$$
$$V(0,\tau)=0,\quad 0\leqslant\tau\leqslant T,$$
$$V(S,\tau)=S-Ee^{-c(T-\tau)},\quad S\to\infty.$$

Note that σ is a parameter that depends on each concrete model, for example, $\sigma^2=\sigma_{JS}^2$ (the Jandacka-Sevcovic model [9]) or $\sigma^2=\sigma_F^2$ (the Frey model [11]), which can be written, respectively, as

$$\sigma_{JS}^2=\sigma_0^2\left(1+\mu(SV_{SS})^{\frac{1}{3}}\right),\quad \sigma_F^2=\frac{\sigma_0^2}{1-\rho SV_{SS}},$$

where $\mu=3(C^2M/(2\pi))^{1/3}$, σ_0 is the volatility of the underlying asset, $M\geq 0$ is the transaction cost measure, $C\geq 0$ is the risk premium measure, and $\rho\geq 0$ is a parameter measuring the market liquidity. Using the change of independent variables $x=\ln(S/E)$, where $x\in\mathbb{R}$, $t=T-\tau$, $t\in(0,T)$ and substituting $u(x,t)=SV_{SS}$ in (3) for the two above models, we obtain problem (1)-(2). Then the function $\beta(u)$ and the initial condition $u_0(x)$ for the corresponding models will also become [9, 10]:

$$\beta_{JS}=\frac{\sigma_0^2}{2}\left(1+\mu(u)^{1/3}\right)u,\quad u_0(x)=\delta(x),$$

$$\beta_F=\frac{\sigma_0^2}{2}\frac{u}{(1-\rho u)^2},\quad u_0(x)=\delta(x),$$

where $\delta(x)$ is the delta function.

In order to find the approximate solution of problem (1)-(2), we must restrict it to a finite spatial interval $x\in(-L,L)$, where $L>0$ is a sufficiently large number. Since $S=Ee^x$, we limit the interval $S\in(0,+\infty)$ by the interval $S\in(Ee^{-L},Ee^{L})$. In practical calculations, we can choose $L\approx 1.5$ to include important values of S. Thus, instead of (2), we consider the Gamma equation (1) with Dirichlet boundary conditions at $x=\pm L$ [9], i.e.,

$$u(-L,t)=u(L,t)=0,\quad u(x,0)=u_0(x). \quad (4)$$

Let $u(x,t)$ be a solution of problem (1)-(2), and let $\bar{D}_u=[m_1,m_2]$ be a segment containing a set of its values, where $m_1\leqslant u(x,t)\leqslant m_2$. If the function $\beta(u)\in C^3(\bar{D}_u)$ for $u\in\bar{D}_u$ is sufficiently smooth, then Eq. (1) can be written as

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+r(u)\frac{\partial u}{\partial x}, \quad (5)$$

with coefficients

$$k(u)=\beta'(u),\quad r(u)=k(u)+c. \quad (6)$$

We assume that the parabolicity condition of equation (5) on the solution [12] is satisfied such that

$$0<k_1\leqslant k(u)\leqslant k_2,\quad \forall u\in\bar{D}_u, \quad (7)$$

where k_1, k_2 are constants chosen based on each model, and

$$\bar{D}_u=\{u(x,t):\ m_1\leqslant u(x,t)\leqslant m_2,\ (x,t)\in\bar{Q}_T\}.$$
$$\bar{Q}_T=\{(x,t):\ -L\leqslant x\leqslant L,\ 0\leqslant t\leqslant T\}.$$

We assume in what follows that there exists a unique solution for problem (1)-(2) and all the coefficients in Eq. (5). We further assume the desired function to have continuous bounded derivatives of the required order as the presentation proceeds.

Using the technique from [13], we prove two-sided estimates for the exact solution of problem (1)-(2).

Theorem 1: let condition (7) be satisfied. Then, for the solution $u(x,t)$ of the problem (1)-(2), the following two-sided estimates are true:

$$m_1 = \min\left\{0, e^T \min_{-L\leqslant x\leqslant L} u_0(x)\right\} \leqslant u(x,t) \leqslant \max\left\{0, e^T \max_{-L\leqslant x\leqslant L} u_0(x)\right\} = m_2 \quad (8)$$

Proof: to prove (8), we make a transformation of the function $u(x, t)$ to the new function $v(x, t)$ associated with the equality

$$u(x,t) = v(x,t)e^{\lambda t},$$

where λ is an arbitrary number. The function $v(x, t)$ satisfies the equation

$$\frac{\partial v}{\partial t} + \lambda v - k\left(ve^{\lambda t}\right)\frac{\partial^2 v}{\partial x^2} - \frac{\partial k\left(ve^{\lambda t}\right)}{\partial x}\frac{\partial v}{\partial x} - r\left(ve^{\lambda t}\right)\frac{\partial v}{\partial x} = 0, \quad (9)$$

with initial and boundary conditions

$$v(x,0) = u_0(x), \quad -L \leqslant x \leqslant L, \quad (10)$$

$$v(-L,t) = v(L,t) = 0, \quad t > 0. \quad (11)$$

Let the maximum of the solution, $v(x, t)$, of problem (9)-(11) be reached at some point $(x_0, t_0) \in (-L, L) \times (0, T]$

$$\max_{(x,t)\in \bar{Q}_T} v(x,t) = v(x_0,t_0),$$

moreover, at the point (x_0, t_0), Eq. (9) and the following relations are satisfied

$$\frac{\partial v(x_0,t_0)}{\partial t} \geqslant 0, \quad \frac{\partial v(x_0,t_0)}{\partial x} = 0,$$
$$\frac{\partial^2 v(x_0,t_0)}{\partial x^2} = \lim_{\Delta x\to 0} \frac{v(x_0-\Delta x,t_0) - 2v(x_0,t_0) + v(x_0+\Delta x,t_0)}{\Delta x^2} \leqslant 0.$$

It follows that $v(x,t) \leqslant v(x_0,t_0) \leqslant 0, \quad \lambda > 0.$ (12)

If the maximum in Q_T value $v(x, t)$ is taken at the boundary $\{-L, L\} \times (0, T] \cup [-L, L] \times \{0\}$, then we obtain

$$v(x,t) \leqslant \max_{(x,t)\in \bar{Q}_T} v(x,t) = \max\left\{0, \max_{-L\leqslant x\leqslant L} u_0(x)\right\}. \quad (13)$$

Thus, in all cases of Eqs. (12)-(13), the following estimate is valid

$$v(x,t) \leqslant \max\left\{0, \max_{-L\leqslant x\leqslant L} u_0(x)\right\},$$

from which it follows

$$u(x,t) \leqslant \max\left\{0, e^T \max_{-L\leqslant x\leqslant L} u_0(x)\right\}, \quad \lambda = 1.$$

The case of the minimum of the solution $u(x, t)$ is proved similarly. Thus, the theorem is proved.

Finite-difference scheme on non-uniform spatial grids

We introduce an arbitrary non-uniform spatial grid

$$\hat{\bar{\omega}} = \hat{\omega}_h \cup \gamma_h, \quad \hat{\omega} = \{x_i = x_{i-1} + h_i, i = 1,2,\ldots,N-1\}, \quad \gamma_h = \{x_0 = -L, x_N = L\},$$

and uniform grid by the time variable

$$\bar{\omega}_\tau = \{t_n = n\tau, \quad n = 0,1,\ldots,N_0, \quad \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

Taking into account the identity $(ku')' = 0.5((ku)'' + ku'' - k''u)$ and using standard notation [1]

$$h_+ = h_{i+1}, \quad h = h_i, \quad \hbar = (h_+ + h)/2, \quad v = v_i = v(x_i), \quad v_\pm = v_{i\pm 1} = v(x_{i\pm 1}),$$

$$v_x = (v_+ - v)/h_+, \quad v_{\bar{x}} = (v - v_-)/h, \quad v_{\bar{x}\hat{x}} = (v_x - v_{\bar{x}})/\hbar,$$
$$t = t_n, \quad \hat{t} = t_{n+1}, \quad v = v^n = v(t_n), \quad \hat{v} = v^{n+1} = v(t_{n+1}),$$

we construct a difference scheme for a quasilinear parabolic equation (5) on a non-uniform grid

$$\omega = \hat{\omega}_h \times \omega_\tau,$$
$$y_{t(\beta_1\beta_2)} = \kappa(y)A_1\hat{y} + b^+(y)a_+(y)\hat{y}_x + b^-(y)a(y)\hat{y}_{\bar{x}}, \quad (14)$$
$$y_i^0 = u_0(x_i), \quad y_0^{n+1} = y_N^{n+1} = 0,$$

where

$$v_{(\beta_k\beta_{k+1})} = \beta_k v_+ + (1 - \beta_k - \beta_{k+1})v + \beta_{k+1}v_-,$$
$$A_1\hat{y} = 0.5\left[(k(y)\hat{y})_{\bar{x}\hat{x}} + k_{(\beta_1\beta_2)}(y)\hat{y}_{\bar{x}\hat{x}} - k_{\bar{x}\hat{x}}(y)\hat{y}_{(\beta_3\beta_4)}\right],$$
$$\beta_1 = 0.5\left(|\tilde{h}| + \tilde{h}\right)/h_+, \quad \beta_2 = 0.5\left(|\tilde{h}| - \tilde{h}\right)/h, \quad \tilde{h} = (h_+ - h)/3,$$
$$\beta_3 = 0.5\left(\tilde{h}k_{\bar{x}\hat{x}} - |\tilde{h}k_{\bar{x}\hat{x}}|\right)/(h_+k_{\bar{x}\hat{x}}), \quad \beta_4 = -0.5\left(\tilde{h}k_{\bar{x}\hat{x}} + |\tilde{h}k_{\bar{x}\hat{x}}|\right)/(hk_{\bar{x}\hat{x}}),$$
$$b^\pm(y) = \frac{1}{3}\left(\frac{r^\pm}{k}(y_-) + \frac{r^\pm}{k}(y) + \frac{r^\pm}{k}(y_+)\right), \quad r^\pm(y) = 0.5(r(y) \pm |r(y)|),$$
$$\kappa(y) = \frac{1}{1 + R(y)}, \quad R(y) = \frac{h_+ + 2h}{6}b^+(y) - \frac{2h_+ + h}{6}b^-(y) \geqslant 0,$$
$$a(y) = 0.5\left(k(y) + k(y_-)\right), \quad a_+(y) = 0.5\left(k(y_+) + k(y)\right).$$

Approximation error: let us prove that the scheme (14) approximates problem (5) under the conditions of Eq. (4) in the second order with respect to the point $\bar{x}_i = x_i + \tilde{h}_i$ (in the case of a uniform grid $\bar{x}_i = x_i$). To do this, we focus on the relationship [7]

$$v_{\bar{x}\hat{x}} - v''(\bar{x}) = O(\hbar^2), \quad (15)$$

$$v_{(\beta_k\beta_{k+1})} - v(\bar{x}) = O(\hbar^2), \quad k \in \{1,3\}, \quad (16)$$

when the condition of variable in space weight factors is fulfilled β_k, β_{k+1},

$$\beta_k h_+ - \beta_{k+1}h = \frac{h_+ - h}{3} = \tilde{h}, \quad k \in \{1,3\}.$$

By virtue of (15)

$$(k(u)\hat{u})_{\bar{x}\hat{x}} - \frac{\partial^2(k(u)u)(\bar{x},\hat{t})}{\partial x^2} = O(\hbar^2 + \tau), \quad k_{\bar{x}\hat{x}}(u) - \frac{\partial^2 k(\bar{x})}{\partial x^2} = O(\hbar^2). \quad (17)$$

In view of (16) we obtain

$$k_{(\beta_1\beta_2)}(u) - k(\bar{x}) = O(\hbar^2), \quad (18)$$

$$u_{t(\beta_1\beta_2)} - \frac{\partial u(\bar{x},\hat{t})}{\partial t} = O(\hbar^2 + \tau). \quad (19)$$

From (15)-(18), it follows that

$$A_1\hat{u} - \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)(\bar{x},\hat{t}) = O(\hbar^2 + \tau). \quad (20)$$

Using the Taylor series expansion

$$u_x = u'(\bar{x}) + \frac{h_+ + 2h}{6}u''(\bar{x}) + O(\hbar^2), \quad u_{\bar{x}} = u'(\bar{x}) - \frac{2h_+ + h}{6}u''(\bar{x}) + O(\hbar^2),$$
$$a_+(u) = k(\bar{x}) + \frac{h_+ + 2h}{6}k'(\bar{x}) + O(\hbar^2), \quad a(u) = k(\bar{x}) - \frac{2h_+ + h}{6}k'(\bar{x}) + O(\hbar^2),$$

we conclude that

$$a_+(u)u_x = (ku')(\bar{x}) + \frac{h_+ + 2h}{6}(ku')'(\bar{x}) + O(\hbar^2),$$

$$a(u)u_{\bar{x}} = (ku')(\bar{x}) - \frac{2h_+ + h}{6}(ku')'(\bar{x}) + O(\hbar^2).$$

Since

$$r^+(u) + r^-(u) = r(u),$$

$$b^+(u) + b^-(u) = \frac{1}{3}\left(\frac{r}{k}(u_-) + \frac{r}{k}(u) + \frac{r}{k}(u_+)\right) = \frac{r}{k}(\bar{x}) + O(\hbar^2),$$

then

$$b^+(u)a_+(u)u_x + b^-(u)a(u)u_{\bar{x}} = (ru')(\bar{x}) + R(u)(ku')'(\bar{x}) + O(\hbar^2). \quad (21)$$

Using (21) we get

$$b^+(u)a_+(u)\hat{u}_x + b^-(u)a(u)\hat{u}_{\bar{x}} = \left(r(u)\frac{\partial u}{\partial x}\right)(\bar{x},\hat{t}) + R(u)\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)(\bar{x},\hat{t}) + O(\hbar^2 + \tau). \quad (22)$$

Finally, from (19)–(20), (22) we find out that the approximation error is of second order in space

$$\psi(\bar{x},\hat{t}) = -u_{t(\beta_1\beta_2)} + \kappa(u)A_1\hat{u} + b^+(u)a_+(u)\hat{u}_x + b^-(u)a(u)\hat{u}_{\bar{x}}$$

$$= \frac{R^2(u)}{1+R(u)}\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)(\bar{x},\hat{t}) + O(\hbar^2 + \tau) = O(\hbar^2 + \tau).$$

Therefore, spatial approximation order of the difference scheme (14) is 2 and its temporal approximation order is 1.

Monotonicity, two-sided and a priori estimates

We write the difference scheme (14) in the canonical form [2]

$$A_i^n y_{i-1}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+1}^{n+1} = -F_i^n, i = 1,2,\dots,N-1, \quad (23)$$

$$y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \quad (24)$$

with coefficients defined as follows

$$A_i^n = -\beta_{2i} + 0.5\kappa_i^n\tau\left[\left(k_{(\beta_1\beta_2)}(y^n) + k(y_{i-1}^n)\right)/(\hbar_i h_i) - \beta_{4i}k_{\bar{x}\hat{x},i}(y^n)\right] - \tau b_i^-(y^n)a_i^n/h_i,$$

$$B_i^n = -\beta_{1i} + 0.5\kappa_i^n\tau\left[\left(k_{(\beta_1\beta_2)}(y^n) + k(y_{i+1}^n)\right)/(\hbar_i h_{i+1}) - \beta_{3i}k_{\bar{x}\hat{x},i}(y^n)\right] + \tau b_i^+(y^n)a_{i+1}^n/h_{i+1},$$

$$C_i^n = 1 + A_i^n + B_i^n, \quad F_i^n = y_{(\beta_1\beta_2)}^n, \quad \mu_1(t_{n+1}) = \mu_2(t_{n+1}) = 0.$$

The scheme (23)-(24) is monotone if the positivity conditions of the coefficients are satisfied [1], i.e.

$$A_i^n > 0, \quad B_i^n > 0, \quad D_i^n = C_i^n - A_i^n - B_i^n > 0. \quad (25)$$

Base on the maximum principle, similiar to the work of [14], we formulate the following results for the difference schemes (14):

Theorem2(Maximumprinciple):letpositivityconditions for the coefficients in Eq. (25) be fulfilled. Then, for the solution of the difference scheme given by Eqs. (23)-(24), the following two-sided estimate is valid:

$$\min\left\{\mu_1^{n+1}, \mu_2^{n+1}, \min_{1\leqslant i\leqslant N-1}\frac{F_i^n}{D_i^n}\right\} \leqslant y_i^{n+1} \leqslant \max\left\{\mu_1^{n+1}, \mu_2^{n+1}, \max_{1\leqslant i\leqslant N-1}\frac{F_i^n}{D_i^n}\right\}, \quad i = \overline{0,N}. \quad (26)$$

Proof: suppose that the maximum of the solution, *y(x)*, of the difference problem (23)-(24) is reached on the boundary point such that

$$y_i^{n+1} \leqslant \max_{0\leqslant i\leqslant N} y_i^{n+1} = \max\{\mu_1^{n+1}, \mu_2^{n+1}\}. \quad (27)$$

If the grid function, *y(x)*, reaches its maximum at an interior grid-point x_{i^*}, $1 \leqslant i^* \leqslant N-1$, then

$$C_{i^*}^n y_{i^*}^{n+1} = A_{i^*}^n y_{i^*-1}^{n+1} + B_{i^*}^n y_{i^*+1}^{n+1} + F_{i^*}^n \leqslant (A_{i^*}^n + B_{i^*}^n)y_{i^*}^{n+1} + F_{i^*}^n.$$

In view of the conditions of the theorem $D_{i^*}^n = C_{i^*}^n - A_{i^*}^n - B_{i^*}^n > 0$, we have

$$y_i^{n+1} \leqslant \max_{0\leqslant i\leqslant N} y_i^{n+1} = y_{i^*}^{n+1} \leqslant \frac{F_{i^*}^n}{D_{i^*}^n} \leqslant \max_{1\leqslant i\leqslant N-1}\frac{F_i^n}{D_i^n}. \quad (28)$$

From Eqs. (27)-(28) we obtain the right-hand side of the estimate in Eq. (26). In a similar way, the lower bound can be proved. The theorem is proved.

Now we need to find a condition such that $y_i^n \in \bar{D}_u$ for all *i, n*. When *n = 0*, it is obvious that $y_i^0 = u_0(x_i) \in \bar{D}_u$. Assume that, for any arbitrary n, $y_i^n \in \bar{D}_u$, is also true for all i. From this assumption for the case of $\tilde{h} > 0$, $k_{\bar{x}\hat{x}} > 0$, (we do not consider trivial cases of $\tilde{h} = 0$ and $k_{\bar{x}\hat{x}} = 0$) we obtain the following concrete values of the weights

$$\beta_1 = \tilde{h}/h_+ > 0, \quad \beta_2 = \beta_3 = 0, \quad \beta_4 = -\tilde{h}/h < 0,$$

$$k_{(\beta_1\beta_2)}(y) = \frac{\tilde{h}}{h_+}k(y_+) + \left(1 - \frac{\tilde{h}}{h_+}\right)k(y) = \frac{\tilde{h}}{h_+}k(y_+) + \frac{2h_+ + h}{3h_+}k(y) > 0,$$

$$-\beta_4 k_{\bar{x}\hat{x}}(y) = \frac{\tilde{h}}{h}k_{\bar{x}\hat{x}}(y) > 0.$$

It follows that $A_i^n > 0$. It is easy to show that $B_i^n > 0$ at $\tau \geqslant (1 + 0.5\bar{h}c_0)\left|h_{i+1}^2 - h_i^2\right|/(6k_1)$, $\bar{h} = \max_{1\leqslant i\leqslant N} h_i$, and and $c_0 = \max_{u\in\bar{D}_u}\frac{|r(u)|}{k(u)}$. In a similar way we can investigate all the other cases.

Therefore, the inequality

$$\tau \geqslant \frac{(1+0.5\bar{h}c_0)\left\|h_+^2 - h^2\right\|_C}{6k_1}, \quad \bar{h} = \max_{1\leqslant i\leqslant N} h_i, \quad c_0 = \max_{u\in\bar{D}_u}\frac{|r(u)|}{k(u)}, \quad (29)$$

guarantees the fulfilment of the positivity of the coefficients of Eq. (25) (i.e. the difference scheme (14) is monotone). According to Theorem 2 on the basis of the estimate of Eq. (26) for arbitrary $t = t_n \in \omega_\tau$ and all $i = 0,1,\dots, N$, we have

$$\min\left\{0, \min_{1\leqslant i\leqslant N-1} y_{(\beta_1\beta_2)}^n\right\} \leqslant y_i^{n+1} \leqslant \max\left\{0, \max_{1\leqslant i\leqslant N-1} y_{(\beta_1\beta_2)}^n\right\}. \quad (30)$$

With the help of inequalities $\max\limits_{1\leqslant i\leqslant N-1} y^n_{(\beta_1\beta_2)} \leqslant \max\limits_{1\leqslant i\leqslant N-1} y^n_i$, $\min\limits_{1\leqslant i\leqslant N-1} y^n_i \leqslant \min\limits_{1\leqslant i\leqslant N-1} y^n_{(\beta_1\beta_2)}$ (since variable weight factors β_1, β_2, are non-negative) from Eq. (30) we have

$$\min\left\{0, \min_{0\leqslant i\leqslant N} y^n_i\right\} \leqslant y^{n+1}_i \leqslant \max\left\{0, \max_{0\leqslant i\leqslant N} y^n_i\right\}. \tag{31}$$

Using induction on n, according to Eq. (31) we acquire the two-sided estimate of the difference solution via the input data without assumption of its sign-definiteness

$$m_1 \leqslant \min\left\{0, \min_{-L\leqslant x\leqslant L} u_0(x)\right\} \leqslant y^{n+1}_i \leqslant \max\left\{0, \max_{-L\leqslant x\leqslant L} u_0(x)\right\} \leqslant m_2, \quad i = 0,1,\dots N. \tag{32}$$

In view of Eq. (32) we conclude that $y^{n+1}_i \in \bar{D}_u$ for all $i = \overline{0,N}$. Therefore, the following theorem is proved.

Theorem 3: let the conditions of Eq. (29) be fulfilled. Then, the finite-difference scheme of Eq. (14) is monotone, and its solution belongs to the value interval of the exact solution $y \in \bar{D}_u$ and the above two-sided estimates of Eq. (32) hold.

With the help of the maximum principle we acquire the a priori estimate of the solution of the difference scheme of Eq. (14) in the C-norm:

Theorem 4: let the condition of Theorem 3 be fulfilled. Then, for the solution of the difference scheme of Eq. (14), the following a priori estimate holds

$$\|y^n\|_{\bar{C}} \leqslant \|u_0\|_{\bar{C}}. \tag{33}$$

Remark 1: note that the maximum and minimum values of the difference solution do not depend on the diffusion coefficient nor the convection coefficient.

Remark 2: the estimates obtained in Eq. (32) are fully consistent with the estimates of the exact solution of the differential problem given by Eq. (8).

Remark 3: if the grid is uniform in space ($h_+ = h$), then the scheme given by Eq. (14) is transformed into the well-known purely implicit scheme:

$$y_t = \kappa(y)(a(y)\hat{y}_{\bar{x}})_x + b^+(y)a_+(y)\hat{y}_x + b^-(y)a(y)\hat{y}_{\bar{x}},$$

for which the a priori estimates of Eqs. (32)-(33) have already been fulfilled without the restrictions of Eq. (29) on the relation between the grid steps (unconditional monotonicity). Here:

$$\kappa(y) = \left(1 + R(y)\right)^{-1}, \quad R(y) = 0.5h|r(y)|/k(y),$$
$$a(y) = 0.5[k(y) + k(y_-)], \quad b^{\pm}(y) = (r^{\pm}/k)(y).$$

Numerical implementation

Because the Gamma equation has no exact solutions (only analytical solutions), to assess the efficiency of the proposed difference scheme and to maintain the equality of the Gamma equation, we must add a residual term $f(x, t)$ to the right-hand side of Eq. (5). We consider Eq. (5) in the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + r(u)\frac{\partial u}{\partial x} + f(x,t), \tag{34}$$

with the boundary conditions:

$$u(-\pi, t) = u(\pi, t) = 0, \quad u(x, 0) = u_0(x), \tag{35}$$

and input data:

$$k(u) = u^2 + 1, \quad r(u) = \sqrt{u+4}, \quad u = u(x,t), \quad x \in [-\pi, \pi], \quad T = 0.5,$$
$$f(x,t) = \frac{1}{4}e^t\left((8 + e^{2t})\sin(x) - 4\cos(x)\sqrt{e^t \sin(x)} - 3e^{2t}\sin(3x)\right), u_0(x) = \sin(x),$$

and suppose that an exact solution is $u(x, t) = e^t \sin(x)$.

Obviously, we have $-e^{0.5} \leq u(x, t) \leq e^{0.5}$, i.e. $m_1 = -e^{0.5}$, $m_2 = e^{0.5}$. Then for all $u \in [-e^{0.5}, e^{0.5}]$ we obtain $0 < 1 \leq k(u) \leq e + 1$, and according to the condition of Eq. (7), Eq. (34) is parabolic (Fig. 1).

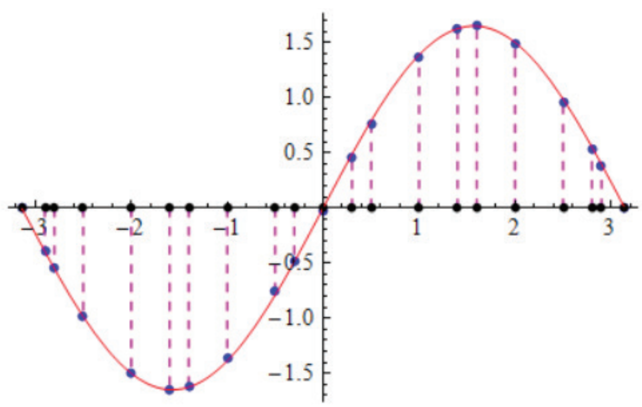

Fig. 1. Exact (red line) and approximate (blue nodes) solutions of the problem (34), (35) at $T = 0.5$ with $\tau = 0.01$.

In Table 1 we show the non-uniform spatial nodes and the error of the method in maximum norm

$$\|z\|_C = \|y - u\|_C = \max_{(x,t)\in\omega} |y(x,t) - u(x,t)|$$

for the difference scheme given in Eq. (14). The approximate solution of the problem given by Eqs. (34), (35) at t = 0.5, obtained by the difference scheme in Eq. (14), is shown on Fig. 1.

The computational experiment illustrates the higher accuracy of the new scheme on coarse space grids. For the scheme of Eq. (14) the accuracy of order $O(h^2 + \tau)$ is reached on the coarse grids.

Table 1. Numerical results on non-uniform spatial grids for problem (34), (35) at $T = 0.5$ with $\tau = 0.01$.

x_i	$-\pi$	-2.9	-2.8	-2.5	-2	-1.6	-1.4	-1	-0.5	-0.3	
$\|\|Z\|\|_C$	0	0.009	0.009	0.009	0.001	0.003	0.0004	0.01	0.02	0.01	
x_i	0	0.3	0.5	1	1.4	1.6	2	2.5	2.8	2.9	π
$\|\|Z\|\|_C$	0.02	0.03	0.02	0.01	0.001	0.0001	0.008	0.02	0.02	0.015	0

Conclusions

Problems requiring a solution to nonlinear partial differential equations arise in elasticity theory, financial mathematics, physical chemistry, biology, and other fields. The demand to solve these problems has caused rapid development of numerical methods for their solution. By virtue of its comparative simplicity and versatility, the finite difference method is often used.

In the present paper we proposed a new second-order in a space monotone difference scheme on a non-uniform grid that approximates the Dirichlet IBVP for a quasi-linear parabolic equation, namely, the one-dimensional non-linear Gamma equation in financial mathematics. Under several constraints on the grid, two-side estimates of the solution of the scheme are established. Note that the proven two-side estimates of difference solution are fully consistent with estimates of the solution of the differential problem. Moreover, the maximum and minimum values of the difference solution are not dependent on the diffusion and convection coefficients.

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