DOI: 10.31276/VJSTE.2026.0007 ***Mathematics and Computer Science*** | *Mathematics; Computer Science; Computational Science*

Operations, similarity measures, and rough-set approximations for uncertain sets

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Received 24 January 2026; revised 29 January 2026; accepted 1 May 2026

Abstract:

An uncertain set assigns a generalised uncertainty value to each element, thus providing a single membership-function template that encompasses many degree-based models, including fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic sets [1]. Building on this unified viewpoint, we investigate three core components of uncertainty-aware reasoning: algebraic operations, similarity measures, and rough-set-style approximations. First, for a fixed uncertainty model M whose degree domain is equipped with a De Morgan lattice structure, inclusion, complement, union, intersection, difference, and level-set (α-cut) constructions are defined for M-type uncertain sets. Second, the author introduces an axiomatic notion of an uncertain similarity measure and provide concrete families, including normalised L_1- and Chebyshev-type similarities, together with basic properties and reductions to familiar formulas in classical special cases. Third, uncertain rough sets are defined via M-valued relations and corresponding lower/upper approximation operators, and the resulting framework is shown to subsume standard fuzzy rough sets and component-wise neutrosophic rough sets. Finally, a categorical extension based on functorial sets is developed, obtaining functorial rough sets whose approximation operators satisfy naturality with respect to morphisms, thereby enabling the compositional transport of rough approximations across changing universes.

Keywords: **functorial set, fuzzy set, neutrosophic set, rough approximation, similarity measure, uncertain set.**

Classification numbers: **1.1, 1.2, 1.3**

1. Introduction

1.1. Uncertain sets

Classical (crisp) set theory provides a precise and widely used language for formal reasoning and mathematical modelling [2]. However, many real-world problems involve vagueness, incomplete information, or conflicting evidence, which have motivated a broad family of generalised set-theoretic frameworks. Prominent examples include fuzzy sets [3-5], intuitionistic fuzzy sets [6], hesitant fuzzy sets [7], picture fuzzy sets [8], single-valued

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neutrosophic sets (NSs) [9-11], double-valued NSs [12], plithogenic sets [13, 14], and soft sets [15-17].

A fuzzy set assigns a membership grade $\mu(x) \in [0,1]$, to each element x, thereby capturing gradual inclusion rather than a binary yes/no decision [5, 18]. NS refines this idea by associating three (typically independent) degrees $T(x), I(x), F(x) \in [0,1]$, interpreted as truth, indeterminacy, and falsity, respectively [9, 19]. Owing to their expressive power, these models have been used in diverse areas such as decision-making [20], robotics and system integration [21], artificial intelligence [22], and neural networks [23, 24].

More recently, uncertain sets (U-Sets) have been proposed as a unifying umbrella under which many degree-based models - including fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, hesitant, and related structures - can be handled within a single formal template [1, 25-27]. In addition, functorial sets provide a categorical layer that organises such assignments across objects and morphisms, enabling systematic and compositional transport of structure [1, 25]. This unification is valuable both conceptually and algorithmically: it offers a common interface for defining operations and building uncertainty-aware methods in decision science and beyond. Table 1 illustrates how representative degree-based models can be realised within the U-Set framework.

Table 1. Uncertain sets as a unifying framework (illustrative cases).

Model	Degree domain	Realisation as a U-Set
Fuzzy set	$[0,1]$	Choose M with $Dom(M) = [0,1]$ and set $\mu_M(x) = \mu(x)$.
Intuitionistic fuzzy set	$[0,1]^2$	Choose M with $Dom(M) = [0,1]^2$ and set $\mu_M(x) = (\mu(x), \nu(x))$.
NS	$[0,1]^3$	Choose M with $Dom(M) = [0,1]^3$ and set $\mu_M(x) = (T(x), I(x), F(x))$.
Plithogenic set	$[0,1]^m$	Choose M with $Dom(M) \subseteq [0,1]^m$ and set $\mu_M(x)$ to the plithogenic degree vector of x.

U-Set research is important because it provides a unified mathematical interface for diverse uncertainty models, enabling consistent operations, similarity measures, and approximation frameworks, and supporting transferable algorithms for decision-making, learning, and knowledge representation.

1.2. Operations, similarity measures, and rough sets

Operations on fuzzy [28], neutrosophic [29], and plithogenic sets have been studied extensively, with many choices of conjunctions, disjunctions, complements, and aggregation rules tailored to specific applications. In parallel, similarity measures quantify the resemblance between two uncertainty-aware objects by mapping a pair to a score, commonly normalised to the interval $[0,1]$. Such measures play a central role in clustering, pattern recognition, and decision-making under uncertainty. Similarity measures have been studied within each of the fuzzy [30, 31], neutrosophic [32-35], and plithogenic set [36, 37] frameworks.

Rough set (RS) theory provides another complementary viewpoint: given an indiscernibility (or similarity) relation, an imprecise target set is approximated by its lower and upper approximations, representing certain and possible membership, respectively [38, 39]. Many extensions are known, including fuzzy rough sets (FRSs) [40] and neutrosophic rough sets (NRSs) [41, 42], and analogous RS constructions have been explored for other generalised set models. These approximation-based methods have proved useful in decision support, data analysis, and several areas of mathematics and computer science.

1.3. Contributions

The developments reviewed above suggest that U-Sets and functorial sets, together with operations, similarity measures, and rough approximations, form a coherent ecosystem for uncertainty-aware mathematics and algorithms. However, a unified treatment that integrates operations, similarity measures, and rough-set structures directly within the U-Set framework is still relatively limited.

To help fill this gap, these three components are investigated in a common setting of U-Sets and closely related generalisations. This perspective yields a unified and interoperable toolkit: it supports consistent theory, facilitates transferable algorithmic design, and strengthens decision-making under heterogeneous forms of uncertainty. In addition, extensions are developed based on functorial sets. In particular, functorial rough sets (FunRSs) provide natural, compositional rough approximations across changing universes, enabling coherent transport of uncertainty reasoning along morphisms.

2. Preliminaries

In this section, the basic concepts and notation that will be used later are collected. Unless stated explicitly, all sets under consideration are assumed to be finite.

2.1. Fuzzy set and neutrosophic set

A fuzzy set attaches to every element of a given universe a membership degree lying in the unit interval $[0,1]$. In this way, gradual levels of membership are used to capture vagueness, instead of a sharp true/false dichotomy [5, 43]. Several notable generalisations of fuzzy sets have been introduced, such as intuitionistic fuzzy sets [44], superhyperfuzzy sets [45], and hesitant fuzzy sets [7]. These frameworks enrich the classical fuzzy paradigm by allowing more flexible descriptions of uncertainty and preference. For completeness, the basic notions, together with their main extensions, are recalled.

Definition 2.1 (Fuzzy set) [5, 46]. Let Y be a non-empty universe. A fuzzy set on Y is a mapping

$$\tau: Y \to [0,1].$$

A fuzzy relation on Y is a fuzzy subset

$$\delta: Y \times Y \to [0,1].$$

We say that δ is a fuzzy relation on τ if, for all $y, z \in Y$,

$$\delta(y,z) \le min\{\tau(y), \tau(z)\}.$$

NSs refine the fuzzy framework by assigning to each element three separate degrees: a truth degree, an indeterminacy degree, and a falsity degree [9, 11, 19, 47]. In addition, several generalisations of NSs have been proposed, including plithogenic sets [13, 14], which further increase the modelling capacity for complex uncertainty. The formal definition is given below.

Definition 2.2 () [19, 48]. Let X be a non-empty set. An NS A on X is determined by three functions

$$T_A: X \to [0,1], \qquad I_A: X \to [0,1], \qquad F_A: X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent, respectively, the degrees of truth, indeterminacy, and falsity of x with respect to A. These degrees satisfy

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

for every $x \in X$.

2.2. Uncertain set and functorial set

A U-Set assigns a generalised uncertainty value to every element, subsuming fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related models [1]. A functorial set is a category-based construction mapping objects to sets and morphisms to structure-preserving functions, respecting composition and identities [1, 25].

Definition 2.3 (U-Set) [1]. Let X be a non-empty universe and let M be a fixed uncertainty model with a degree-domain $Dom(M) \subseteq [0,1]^k$ for some integer $k \ge 1$ (e.g. fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, etc.). A U-Set of type M (or U-Set of type M) on X is a mapping

$$\mu_M: X \to Dom(M).$$

For each element $x \in X$, the value $\mu_M(x) \in Dom(M)$ is called the M-membership degree of x. Classical fuzzy sets, intuitionistic fuzzy sets, NSs, plithogenic sets, and so on, are obtained by choosing M to be the corresponding uncertainty model.

Definition 2.4 (Functorial set) [1]. Let C be a category and let

$$F: C \to Set$$

be a covariant functor. The pair (C, F) is called a functorial set. For each object $X \in Ob(C)$, the set $F(X)$ is the set of F-elements (or F-structures) over X. Every morphism $f: X \to Y$ in C induces a structure-preserving map

$$F(f): F(X) \to F(Y),$$

such that $F(id_X) = id_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms f, g in C.

2.3. Rough set

RS theory describes an imprecise subset using a pair of lower and upper approximations generated from equivalence classes, thereby representing both certain and possible membership of elements [38, 39]. Several extensions have been proposed, including the HyperRough set [45], the weighted rough set [49], the FRS [50], and the NRS [41, 51]. The basic definitions are recalled below.

Definition 2.5 (RS approximation) [52]. Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence (indiscernibility) relation on X. The relation R partitions X into disjoint equivalence classes, and for each $x \in X$, we write:

$$[x]_R = \{ y \in X \mid (x, y) \in R \}.$$

For any subset $U \subseteq X$, the lower and upper approximations of U with respect to R are defined as follows:

- The lower approximation of U is:

$$\underline{U} = \{ x \in X \mid [x]_R \subseteq U \}.$$

Thus, $\underline{U}$ consists of all elements whose entire equivalence class is contained in U; these elements certainly belong to U.

- The upper approximation of U is

$$\overline{U} = \{ x \in X \mid [x]_R \cap U \neq \emptyset \}.$$

Hence, $\overline{U}$ collects all elements whose equivalence class has a non-empty intersection with U; these elements may belong to U.

The pair $(\underline{U}, \overline{U})$ is called the RS determined by U, and always satisfies

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Definition 2.6 (FRS) [53-55]. Let U be a finite, non-empty universe and let R be a fuzzy relation on U. Consider a fuzzy set A on U with membership function $\mu_A: U \to [0,1]$.

The fuzzy rough lower approximation of A with respect to R, denoted $\underline{apr_R}(A)$, is defined for every $x \in U$ by

$$\underline{apr_R}(A)(x) = inf_{y \in U} max(1 - R(x,y), \mu_A(y)), \qquad x \in U.$$

The fuzzy rough upper approximation of A with respect to R, denoted $\overline{apr}_R(A)$, is given by

$$\overline{apr}_R(A)(x) = sup_{y \in U} min(R(x,y), \mu_A(y)), \quad x \in U.$$

The corresponding boundary region of A is

$$bnd_R(A) = \overline{apr}_R(A) - \underline{apr_R}(A).$$

Whenever $bnd_R(A) \neq \emptyset$, the fuzzy set A is called an FRS.

Definition 2.7 (NRS) [41, 42, 56]. Let U be a non-empty universe and let R be a single-valued neutrosophic relation on U. For each pair $(x, y) \in U \times U$, write

$$R(x,y) = (R_T(x,y), R_I(x,y), R_F(x,y)),$$

where $R_T(x,y)$, $R_I(x,y)$, and $R_F(x,y)$ denote the truth, indeterminacy, and falsity degrees, respectively. Let $A \subseteq U$.

The neutrosophic lower approximation of A with respect to R, denoted $\underline{R}(A)$, is the triple of mappings

$$\begin{aligned} \underline{R}_T(A)(x) &= inf_{y\in A} R_T(x,y), \\ \underline{R}_I(A)(x) &= sup_{y\in A} R_I(x,y), \qquad \forall x \in U \\ \underline{R}_F(A)(x) &= sup_{y\in A} R_F(x,y). \end{aligned}$$

The neutrosophic upper approximation of A with respect to R, denoted $\overline{R}(A)$, is defined by

$$\begin{aligned} \overline{R}_T(A)(x) &= sup_{y\in U} min\left(R_T(x,y), 1_{\{y\in A\}}\right), \\ \overline{R}_I(A)(x) &= inf_{y\in U} max\left(R_I(x,y), 1_{\{y\notin A\}}\right), \ \forall x \in U \\ \overline{R}_F(A)(x) &= inf_{y\in U} max\left(R_F(x,y), 1_{\{y\notin A\}}\right). \end{aligned}$$

where $1_{\{y\in A\}}$ (respectively, $1_{\{y\notin A\}}$) denotes the indicator of membership (respectively, non-membership) in A.

The pair $\left(\underline{R}(A), \overline{R}(A)\right)$ is called the NRS of A induced by R.

3. Results

The main findings of this article are presented in this section.

3.1. Basic operations of uncertain sets

In this subsection, we fix an uncertainty model M with a degree-domain $Dom(M) \subseteq [0,1]^k$ for some integer $k \geq 1$. We assume that $Dom(M)$ is equipped with

- a partial order $\leq_M$ (usually the coordinate-wise order),

- a binary "union" operation $\oplus_M: Dom(M) \times Dom(M) \to Dom(M)$,

- a binary "intersection" operation $\otimes_M: Dom(M) \times Dom(M) \to Dom(M)$,

- and a unary "complement" operation $N_M: Dom(M) \to Dom(M)$, such that $(Dom(M), \oplus_M, \otimes_M, N_M)$ is a De Morgan lattice.[1]

Remark 3.1. The De Morgan lattice [57] assumption provides a uniform algebraic setting in which unions, intersections, and complements are realised by the join $\oplus_M$, meet $\otimes_M$, and negation N_M, so that many structural properties (e.g., monotonicity and De Morgan's laws) follow abstractly. We note, however, that this choice restricts the admissible aggregation operators; in particular, common non-idempotent t-norm/t-conorm pairs (e.g., the product t-norm) are not covered unless one moves to a different algebraic base (such as a residuated lattice).

Let X be a non-empty universe and let

$$\mu_A: X \rightarrow Dom(M)$$

be a U-Set of type M on X (an M-membership function).

Definition 3.2 (Inclusion and equality). Let A and B be $uncertain\ sets$ of type M on X with membership functions $\mu_A, \mu_B: X \rightarrow Dom(M)$.

(i) We say that A is *contained* in B (and write $A \subseteq_M B$) if

$$\mu_A(x) \ \leq_M \ \mu_B(x) \quad for\ all\ x \in X.$$

(ii) We say that A and B are equal (and write $A =_M B$) if $A \subseteq_M B$ and $B \subseteq_M A$.

Example 3.3 (Inclusion and equality of U-Sets). Let $X = \{x_1, x_2, x_3\}$ and take the fuzzy model $Dom(M) = [0,1]$ with the usual order, $\oplus_M = max$, $\otimes_M = min$, $N_M(t) = 1 - t$.

Define two U-Sets A, B of type M by

$$\mu_A(x_1) = 0.2,\ \mu_A(x_2) = 0.5,\ \mu_A(x_3) = 0.7,$$

$$\mu_B(x_1) = 0.4,\ \mu_B(x_2) = 0.5,\ \mu_B(x_3) = 0.9.$$

Then,

$$\mu_A(x_i) \leq \mu_B(x_i) \quad (i = 1,2,3),$$

so $A \subseteq_M B$ but $B \nsubseteq_M A$, hence $A \neq_M B$.

Now, define C, D by

$$\mu_C(x_i) = \mu_D(x_i) = \mu_A(x_i) \quad (i = 1,2,3).$$

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[1]For example, when Dom(M)=$[0,1]^k$, we may take $\leq_M$ to be the coordinate-wise order, $\oplus_M$ and $\otimes_M$ to be coordinate-wise max and min, and N_M $(a_1,\ldots,a_k)$=$(1-a_1,\ldots,1-a_k)$.

Then, $C \subseteq_M D$ and $D \subseteq_M C$, so $C =_M D$.

Definition 3.4 (Complement of a U-Set). The complement of a U-Set A of type M is the U-Set A^c of type M on X whose membership function $\mu_{A^c}: X \to Dom(M)$ is defined by

$$\mu_{A^c}(x) \ := \ N_M(\mu_A(x)), \quad x \in X.$$

Example 3.5 (Complement of a U-Set). With X and M as above, let A be given by

$$\mu_A(x_1) = 0.3, \quad \mu_A(x_2) = 0.8, \quad \mu_A(x_3) = 0.1.$$

The complement A^c has membership

$$\mu_{A^c}(x) = N_M(\mu_A(x)) = 1 - \mu_A(x),$$

Hence,

$$\mu_{A^c}(x_1) = 0.7, \quad \mu_{A^c}(x_2) = 0.2, \quad \mu_{A^c}(x_3) = 0.9.$$

Definition 3.6 (Union and intersection). Let A and B be U-Sets of type M on X with membership functions $\mu_A, \mu_B: X \to Dom(M)$.

(i) The union $A \cup_M B$ is the U-Set of type M on X defined by

$$\mu_{A \cup_M B}(x) \ := \ \mu_A(x) \ \oplus_M \ \mu_B(x), \qquad x \in X.$$

(ii) The intersection $A \cap_M B$ is the U-Set of type M on X defined by

$$\mu_{A \cap_M B}(x) \ := \ \mu_A(x) \ \otimes_M \ \mu_B(x), \qquad x \in X.$$

Example 3.7 (Union and intersection of U-Sets). Let X and M be as above, and define A, B by

$$\mu_A(x_1) = 0.2, \quad \mu_A(x_2) = 0.6, \quad \mu_A(x_3) = 0.9,$$

$$\mu_B(x_1) = 0.5, \quad \mu_B(x_2) = 0.4, \quad \mu_B(x_3) = 0.7.$$

Then the union $A \cup_M B$ and intersection $A \cap_M B$ satisfy

$$\mu_{A \cup_M B}(x) = max\{\mu_A(x), \mu_B(x)\}, \qquad \mu_{A \cap_M B}(x) = min\{\mu_A(x), \mu_B(x)\},$$

so

$$\mu_{A \cup_M B}(x_1) = 0.5, \quad \mu_{A \cup_M B}(x_2) = 0.6, \quad \mu_{A \cup_M B}(x_3) = 0.9,$$

$$\mu_{A \cap_M B}(x_1) = 0.2, \quad \mu_{A \cap_M B}(x_2) = 0.4, \quad \mu_{A \cap_M B}(x_3) = 0.7.$$

Definition 3.8 (Difference). Let A and B be U-Sets of type M on X. The difference $A \backslash_M B$ is defined by

$$A \backslash_M B \ := \ A \cap_M B^c.$$

Equivalently, its membership function satisfies

$$\mu_{A\setminus_M B}(x) \;:=\; \mu_A(x) \otimes_M N_M(\mu_B(x)), \qquad x \in X.$$

Example 3.9 (Difference of U-Sets). Let X and M be as above, and let A, B be given by

$$\mu_A(x_1) = 0.9, \quad \mu_A(x_2) = 0.6, \quad \mu_A(x_3) = 0.3,$$

$$\mu_B(x_1) = 0.4, \quad \mu_B(x_2) = 0.7, \quad \mu_B(x_3) = 0.2.$$

Then,

$$\mu_{A\setminus_M B}(x) = \mu_A(x) \otimes_M N_M(\mu_B(x)) = min\{\mu_A(x), 1 - \mu_B(x)\},$$

Hence,

$$\mu_{A\setminus_M B}(x_1) = \{0.9, 0.6\} = 0.6,$$

$$\mu_{A\setminus_M B}(x_2) = \{0.6, 0.3\} = 0.3,$$

$$\mu_{A\setminus_M B}(x_3) = min\{0.3, 0.8\} = 0.3.$$

Definition 3.10 (α–cut of a U-Set). Let A be a U-Set of type M on X and let $c \in Dom(M)$ be a fixed threshold. The α–cut (or level set) of A at level c is the crisp subset

$$[A]_{\geq c} = \{x \in X \mid \mu_A(x) \geq_M c\}.$$

Example 3.11 (α–cut of a U-Set). Let X and M be as above, and consider A with

$$\mu_A(x_1) = 0.2, \quad \mu_A(x_2) = 0.7, \quad \mu_A(x_3) = 0.9.$$

For the threshold $c = 0.6 \in [0,1] = Dom(M)$, the α–cut

$$[A]_{\geq c} = \{x \in X \mid \mu_A(x) \geq c\}$$

is

$$[A]_{\geq 0.6} = \{x_2, x_3\},$$

because $\mu_A(x_1) = 0.2 < 0.6$ while $\mu_A(x_2) = 0.7 \geq 0.6$ and $\mu_A(x_3) = 0.9 \geq 0.6$.

Remark 3.12. If $\leq_M$ is only a partial order, then the α–cut $[A]_{\geq c}$ can depend sensitively on the chosen threshold c; moreover, unlike the totally ordered case, level sets need not form a strictly nested (monotone) family as c varies, since some thresholds may be incomparable under $\leq_M$.

3.2. Similarity measures of uncertain sets

In this subsection, a general notion of similarity for U-Sets is introduced, concrete similarity measures are defined, and simple properties and examples are given. More specialised similarity measures for neutrosophic and related models can be found in the literature; for instance, tangent-type measures for single-valued neutrosophic hypersoft sets.

Definition 3.13 (Uncertain similarity measure). Let X be a finite, non-empty universe and let M be a fixed uncertainty model with degree-domain $Dom(M) \subseteq [0,1]^k$ for some integer $k \geq 1$. Let $U_M(X)$ denote the set of all U-Sets of type M on X, i.e., all mappings $\mu_M: X \to Dom(M)$.

A mapping

$$S: U_M(X) \times U_M(X) \to [0,1]$$

is called an uncertain similarity measure (of type M) if for all U-Sets $A, B, C \in U_M(X)$ the following hold:

$(S1)$ $0 \leq S(A, B) \leq 1$.

$(S2)$ $S(A, B) = 1$ if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

$(S3)$ $S(A, B) = S(B, A)$ (symmetry).

In what follows, we fix a finite universe X with $|X| = n$ and a model M with $Dom(M) \subseteq [0,1]^k$. For each U-Set A of type M we write

$$\mu_A(x) = (\mu_A^1(x), \dots, \mu_A^k(x)) \in [0,1]^k \quad \text{for } x \in X,$$

where $\mu_A^j(x)$ denotes the j-th component of the M-degree of x in A.

Definition 3.14 (L_1-based similarity for U-Sets). Let $A, B \in U_M(X)$. Define the normalised L_1-distance between A and B by

$$d_1(A, B) \ := \ \frac{1}{nk} \sum_{x \in X} \ \sum_{j=1}^{k} \ |\mu_A^j(x) - \mu_B^j(x)|.$$

The L_1-based similarity between A and B is

$$S_1(A, B) \ := \ 1 - d_1(A, B).$$

Example 3.15 (A normalised L^1-based uncertain similarity measure). Assume that the uncertainty domain satisfies

$$Dom(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. For $u = (u_1, \dots, u_k)$, $v = (v_1, \dots, v_k) \in Dom(M)$, define the normalised L^1-distance

$$d_M(u,v) \ := \ \frac{1}{k}\sum_{j=1}^{k} \ \ |u_j - v_j|.$$

Clearly $0 \leq d_M(u,v) \leq 1$ and $d_M(u,v) = 0$ if and only if $u = v$.

For two U-Sets $A, B \in U_M(X)$ with membership functions $\mu_A, \mu_B : X \to Dom(M)$, define

$$S_{L^1}(A,B) \ := \ 1 - \frac{1}{|X|}\sum_{x \in X} \ \ d_M(\mu_A(x), \mu_B(x)).$$

Then, S_{L^1} is an uncertain similarity measure in the sense of Definition 3.13:

- $0 \leq S_{L^1}(A,B) \leq 1$ because each term $d_M(\mu_A(x), \mu_B(x))$ lies in $[0,1]$.

- $S_{L^1}(A,B) = 1$ if and only if $d_M(\mu_A(x), \mu_B(x)) = 0$ for all $x \in X$, i.e. $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

- Symmetry holds since $d_M(u,v) = d_M(v,u)$, hence $S_{L^1}(A,B) = S_{L^1}(B,A)$.

Thus, S_{L^1} provides a concrete example of an uncertain similarity measure for general U-Sets of type M.

Theorem 3.16. For every uncertainty model M and finite universe X, the mapping S_1 in Definition 3.14 is an uncertain similarity measure in the sense of Definition 3.13.

Proof. Let $A, B \in U_M(X)$.

(1) Range. For each $x \in X$ and each $j \in \{1, \ldots, k\}$, we have $\mu_A^j(x), \mu_B^j(x) \in [0,1]$, hence

$$0 \leq |\mu_A^j(x) - \mu_B^j(x)| \leq 1.$$

Summing over all x and j gives

$$0 \ \leq \ \sum_{x \in X} \ \ \sum_{j=1}^{k} \ \ |\mu_A^j(x) - \mu_B^j(x)| \ \leq \ nk.$$

Dividing by $nk > 0$ yields

$$0 \ \leq d_1(A,B) \leq 1.$$

Therefore,

$$0 \leq S_1(A,B) = 1 - d_1(A,B) \leq 1.$$

Thus, (S1) holds.

(2) The identity of indiscernibles. First, if $\mu_A(x) = \mu_B(x)$ for all $x \in X$, then for each x and j we have $\mu_A^j(x) - \mu_B^j(x) = 0$, hence $|\mu_A^j(x) - \mu_B^j(x)| = 0$ and $d_1(A,B) = 0$, so $S_1(A,B) = 1$.

Conversely, suppose $S_1(A,B) = 1$. Then, $d_1(A,B) = 0$. As $d_1(A,B)$ is an average of non-negative terms $|\mu_A^j(x) - \mu_B^j(x)|$, the equality $d_1(A,B) = 0$ implies that every summand is zero, i.e.

$$\left|\mu_A^j(x) - \mu_B^j(x)\right| = 0 \quad \text{for all } x \in X \text{ and all } j.$$

Thus, $\mu_A^j(x) = \mu_B^j(x)$ for all x, j, hence $\mu_A(x) = \mu_B(x)$ for all $x \in X$. So, (S2) holds.

(3) Symmetry. For each x and j, we have

$$|\mu_A^j(x) - \mu_B^j(x)| = |\mu_B^j(x) - \mu_A^j(x)|.$$

Therefore, the sums in $d_1(A,B)$ and $d_1(B,A)$ are identical, so $d_1(A,B) = d_1(B,A)$ and hence $S_1(A,B) = S_1(B,A)$. Thus, (S3) holds.

All three conditions of Definition 3.13 are satisfied, so S_1 is an uncertain similarity measure.

Theorem 3.17 (Reduction to classical similarities). Let X be finite and M an uncertainty model.

(a) If M is the fuzzy model with $Dom(M) = [0,1]$ (so $k = 1$), then for any two fuzzy sets A, B on X, we have $S_1(A,B) = 1 - \frac{1}{|X|}\sum_{x\in X} |\mu_A(x) - \mu_B(x)|$, which is the usual normalised Hamming similarity between fuzzy sets.

(b) If M is the (single-valued) neutrosophic model with $Dom(M) = [0,1]^3$ (truth, indeterminacy, falsity), then for any two NSs A, B on X, we obtain $S_1(A,B) = 1 - \frac{1}{3|X|}\sum_{x\in X} (|T_A(x) - T_B(x)| + |I_A(x) - I_B(x)| + |F_A(x) - F_B(x)|)$, which is the component-wise normalised Hamming-type similarity used in NS theory.

Proof.

(i) If M is the fuzzy model, then $k = 1$ and each degree is a scalar $\mu_A(x) \in [0,1]$. By Definition 3.14,

$$d_1(A,B) = \frac{1}{|X| \cdot 1}\sum_{x\in X} \sum_{j=1}^{1} |\mu_A^1(x) - \mu_B^1(x)| = \frac{1}{|X|}\sum_{x\in X} |\mu_A(x) - \mu_B(x)|.$$

Thus,

$$S_1(A,B) = 1 - d_1(A,B) = 1 - \frac{1}{|X|}\sum_{x\in X} |\mu_A(x) - \mu_B(x)|.$$

(ii) If M is the neutrosophic model, then $k = 3$ and we have $\mu_A(x) = (T_A(x), I_A(x), F_A(x))$. Hence, using Definition 3.14,

$$d_1(A,B) = \frac{1}{|X|\cdot 3}\sum_{x\in X} \sum_{j=1}^{3} |\mu_A^j(x) - \mu_B^j(x)|$$

$$= \frac{1}{3|X|}\sum_{x\in X} (|T_A(x) - T_B(x)| + |I_A(x) - I_B(x)| + |F_A(x) - F_B(x)|),$$

which directly gives the claimed expression for $S_1(A,B) = 1 - d_1(A,B)$.

Definition 3.18 (Chebyshev-type similarity for U-Sets). Let $A, B \in U_M(X)$. Define the Chebyshev (or L_∞) distance between A and B by

$$d_\infty(A,B) \ := \ \max_{x\in X} \max_{1\le j\le k} |\mu_A^j(x) - \mu_B^j(x)|.$$

The corresponding similarity is

$$S_\infty(A,B) \ := \ 1 - d_\infty(A,B).$$

Example 3.19 (Chebyshev-type similarity for neutrosophic U-Sets). Let the universe be

$$X = \{x_1, x_2\},$$

and let M be the neutrosophic uncertainty model with

$$Dom(M) = [0,1]^3,$$

where each triple (T, I, F) represents truth, indeterminacy, and falsity degrees.

Define two U-Sets $A, B \in U_M(X)$ by

$$\mu_A(x_1) = (0.8,0.1,0.1), \quad \mu_A(x_2) = (0.3,0.4,0.3),$$

$$\mu_B(x_1) = (0.7,0.2,0.1), \quad \mu_B(x_2) = (0.4,0.3,0.3).$$

For x_1, we have component-wise absolute differences

$$|\mu_A^1(x_1) - \mu_B^1(x_1)| = |0.8 - 0.7| = 0.1,$$

$$|\mu_A^2(x_1) - \mu_B^2(x_1)| = |0.1 - 0.2| = 0.1,$$

$$|\mu_A^3(x_1) - \mu_B^3(x_1)| = |0.1 - 0.1| = 0,$$

So,

$$\max_{1\leq j\leq 3}|\mu_A^j(x_1) - \mu_B^j(x_1)| = 0.1.$$

For x_2, we obtain

$$|\mu_A^1(x_2) - \mu_B^1(x_2)| = |0.3 - 0.4| = 0.1,$$

$$|\mu_A^2(x_2) - \mu_B^2(x_2)| = |0.4 - 0.3| = 0.1,$$

$$|\mu_A^3(x_2) - \mu_B^3(x_2)| = |0.3 - 0.3| = 0,$$

hence

$$\max_{1\leq j\leq 3}|\mu_A^j(x_2) - \mu_B^j(x_2)| = 0.1.$$

Therefore, the Chebyshev distance is

$$d_\infty(A,B) = \max_{x\in X}\max_{1\leq j\leq 3}|\mu_A^j(x) - \mu_B^j(x)| = \max\{0.1,0.1\} = 0.1,$$

and the corresponding Chebyshev-type similarity is

$$S_\infty(A,B) = 1 - d_\infty(A,B) = 1 - 0.1 = 0.9.$$

Thus, A and B are highly similar in the Chebyshev sense with similarity value 0.9.

Proposition 3.20. For every uncertainty model M and finite universe X, the mapping S_∞ in Definition 3.18 is an uncertain similarity measure.

Proof. As before, for each x and j we have $0 \leq |\mu_A^j(x) - \mu_B^j(x)| \leq 1$, hence

$$0 \leq d_\infty(A,B) \leq 1.$$

Therefore, $0 \leq S_\infty(A,B) = 1 - d_\infty(A,B) \leq 1$, proving (S1). If $\mu_A(x) = \mu_B(x)$ for all x, then every absolute difference is zero and hence $d_\infty(A,B) = 0$, so $S_\infty(A,B) = 1$. Conversely, if $S_\infty(A,B) = 1$, then $d_\infty(A,B) = 0$, so the maximum of all absolute differences is zero; hence, every difference is zero and $\mu_A(x) = \mu_B(x)$ for all x, which proves (S2). Symmetry (S3) is immediate from $|\mu_A^j(x) - \mu_B^j(x)| = |\mu_B^j(x) - \mu_A^j(x)|$.

3.3. Uncertain rough sets

Uncertain rough sets (URSs) generalise rough approximations by using model-valued relations and degrees, yielding lower/upper U-Sets under diverse uncertainty frameworks (cf. [25]). Throughout, we fix an uncertainty model M with degree-domain $Dom(M) \subseteq [0,1]^k$ $(k \geq 1)$ equipped with

$$(Dom(M), \leq_M, \oplus_M, \otimes_M, N_M),$$

where $\leq_M$ is a partial order, $\oplus_M$,$\otimes_M$ are binary operations (generalised union and intersection) and N_M is a complement such that $(Dom(M),\oplus_M,\otimes_M, N_M)$ is a De Morgan lattice. We denote by 0_M and 1_M the least and greatest elements of $Dom(M)$, typically $0_M = (0, \dots ,0)$ and $1_M = (1, \dots ,1)$ with coordinate-wise order.

For a finite universe X, we write $U_M(X)$ for the set of all U-Sets A of type M on X, i.e. all mappings $\mu_A: X \to Dom(M)$.

Definition 3.21 (Uncertain relation of type M). Let X be a non-empty universe. An uncertain relation of type M (or M-relation) on X is a mapping

$$R_M: X \times X \to Dom(M).$$

For $x, y \in X$, the value $R_M(x, y) \in Dom(M)$ is interpreted as the degree to which x and y are indiscernible, similar or related.

Definition 3.22 (Uncertain rough approximations and URS). Let X be a finite non-empty universe, R_M an M-relation on X and $A \in U_M(X)$ a U-Set of type M with membership function $\mu_A: X \to Dom(M)$.

The uncertain rough lower approximation of A with respect to R_M is the U-Set $\underline{R_M}(A) \in U_M(X)$ whose membership function is given by

$$\mu_{\underline{R_M}(A)}(x) \ := \ \wedge_{y \in X}(N_M(R_M(x, y)) \ \oplus_M \ \mu_A(y)), \qquad x \in X,$$

where $\wedge$ denotes the infimum in $(Dom(M), \leq_M)$.

The uncertain rough upper approximation of A with respect to R_M is the U-Set $\overline{R_M}(A) \in U_M(X)$ with membership

$$\mu_{\overline{R_M}(A)}(x) \ := \ \vee_{y \in X}(R_M(x, y) \ \otimes_M \ \mu_A(y)), \qquad x \in X,$$

where $\vee$ is the supremum in $(Dom(M), \leq_M)$.

The ordered pair

$$\left(\underline{R_M}(A), \overline{R_M}(A) \right)$$

is called the URS of A with respect to R_M (or simply a URS of type M).

Remark 3.23. When $Dom(M) \subseteq [0,1]^k$ is ordered coordinate-wise, the infimum/supremum $\wedge, \vee$ are taken component-wise, so the above formulas are explicit coordinate-wise min/max in the usual examples (fuzzy, neutrosophic, plithogenic, etc.).

Theorem 3.24 (Embedding of U-Sets). Let X be finite and $A \in U_M(X)$ a U-Set of type M. Define the identity M-relation

$$R_M^{id}(x, y) := \{1_M, x = y, 0_M, x \neq y, \quad x, y \in X.$$

Then, for this relation,

$$\underline{R_M^{id}}(A) = A \qquad \text{and} \qquad \overline{R_M^{id}}(A) = A.$$

In particular, every U-Set appears as a degenerate URS.

Proof. Fix $x \in X$. For the lower approximation, we have:

$$\mu_{\underline{R_M^{id}}(A)}(x) = \bigwedge_{y \in X}\left(N_M\left(R_M^{id}(x,y)\right) \oplus_M \mu_A(y)\right).$$

If $y = x$, then $R_M^{id}(x,x) = 1_M$ and hence $N_M(1_M) = 0_M$, so

$$N_M\left(R_M^{id}(x,x)\right) \oplus_M \mu_A(x) = 0_M \oplus_M \mu_A(x) = \mu_A(x),$$

using that 0_M is the neutral element for $\oplus_M$.

If $y \neq x$, then $R_M^{id}(x,y) = 0_M$ and $N_M(0_M) = 1_M$, so

$$N_M\left(R_M^{id}(x,y)\right) \oplus_M \mu_A(y) = 1_M \oplus_M \mu_A(y) = 1_M,$$

because 1_M is the greatest element of $Dom(M)$.

Thus, the family over which we take the infimum consists of the single value $\mu_A(x)$ and several copies of 1_M, i.e.

$$\{\mu_A(x)\} \cup \{1_M \ (several\ times)\}.$$

Since $\mu_A(x) \leq_M 1_M$, the infimum is

$$\mu_{\underline{R_M^{id}}(A)}(x) = \mu_A(x).$$

For the upper approximation, we obtain

$$\mu_{\overline{R_M^{id}}(A)}(x) = \bigvee_{y \in X}\left(R_M^{id}(x,y) \otimes_M \mu_A(y)\right).$$

If $y = x$, then

$$R_M^{id}(x,x) \otimes_M \mu_A(x) = 1_M \otimes_M \mu_A(x) = \mu_A(x),$$

while for $y \neq x$,

$$R_M^{id}(x,y) \otimes_M \mu_A(y) = 0_M \otimes_M \mu_A(y) = 0_M.$$

Thus, we take the supremum of $\mu_A(x)$ and several copies of 0_M, so

$$\mu_{\overline{R_M^{id}}(A)}(x) = \mu_A(x).$$

Since $x \in X$ was arbitrary, both equalities follow.

Theorem 3.25 (URSs generalise FRSs). Let M_F be the fuzzy model with

$$Dom(M_F) = [0,1], \quad \leq_{M_F} = \leq, \quad \oplus_{M_F} = max, \quad \otimes_{M_F} = min, \quad N_{M_F}(t) = 1 - t.$$

Let $R: X \times X \to [0,1]$ be a fuzzy relation and $\mu_A: X \to [0,1]$ a fuzzy set. View R as an M_F-relation and A as a U-Set of type M_F. Then, the uncertain rough lower and upper approximations satisfy

$$\mu_{\underline{R_{M_F}}(A)}(x) = inf_{y\in X} max(1 - R(x,y), \mu_A(y)),$$

$$\mu_{\overline{R_{M_F}}(A)}(x) = sup_{y\in X} min\big(R(x,y), \mu_A(y)\big),$$

which are exactly the classical fuzzy rough lower and upper approximations of A with respect to R.

Proof. For the lower approximation in the URS framework, we have

$$\mu_{\underline{R_{M_F}}(A)}(x) = \bigwedge_{y\in X}\big(N_{M_F}(R(x,y)) \ \oplus_{M_F} \ \mu_A(y)\big).$$

By the choice of the fuzzy model,

$$N_{M_F}(R(x,y)) = 1 - R(x,y), \quad a \oplus_{M_F} b = max(a,b),$$

and the infimum is the usual infimum in $[0,1]$. Hence,

$$\mu_{\underline{R_{M_F}}(A)}(x) = inf_{y\in X} max(1 - R(x,y), \mu_A(y)),$$

which is the standard fuzzy rough lower approximation.

Similarly, for the upper approximation,

$$\mu_{\overline{R_{M_F}}(A)}(x) = \bigvee_{y\in X}\big(R(x,y) \ \otimes_{M_F} \ \mu_A(y)\big) = sup_{y\in X} min(R(x,y), \mu_A(y)),$$

because $\otimes_{M_F}$ is the usual min and $\bigvee$ is the supremum in $[0,1]$. These coincide with the classical fuzzy rough upper approximation. Thus, the URS construction reduces to the FRS construction when $M = M_F$.

Theorem 3.26 (URSs generalise NRSs). Let M_{NS} be the single-valued neutrosophic model with $Dom(M_{NS}) = [0,1]^3$, where each degree is a triple (T, I, F) (truth, indeterminacy, falsity). Equip $Dom(M_{NS})$ with component-wise order and operations

$$(T, I, F) \oplus_{M_{NS}} (T', I', F') := \ (max(T, T'), max(I, I'), max(F, F')),$$

$$(T, I, F) \otimes_{M_{NS}} (T', I', F') := \ (min(T, T'), min(I, I'), min(F, F')),$$

$$N_{M_{NS}}(T, I, F) := \ (1 - T, \ 1 - I, \ 1 - F).$$

Let $R_{NS}: X \times X \to [0,1]^3$ be a neutrosophic relation and let A be a single-valued NS on X with membership functions

$$\mu_A(x) = (T_A(x), I_A(x), F_A(x)).$$

Viewing R_{NS} as an M_{NS}-relation and A as a U-Set of type M_{NS}, the uncertain rough lower and upper approximations $\underline{R_{NS}}(A)$ and $\overline{R_{NS}}(A)$ are NSs whose components are given by

$$T_{\underline{R_{NS}}(A)}(x) = inf_{y\in X} max\big(1 - T_{R_{NS}}(x,y),\, T_A(y)\big),$$

$$I_{\underline{R_{NS}}(A)}(x) = inf_{y\in X} max\big(1 - I_{R_{NS}}(x,y),\, I_A(y)\big),$$

$$F_{\underline{R_{NS}}(A)}(x) = inf_{y\in X} max\big(1 - F_{R_{NS}}(x,y),\, F_A(y)\big),$$

and

$$T_{\overline{R_{NS}}(A)}(x) = sup_{y\in X} min\big(T_{R_{NS}}(x,y),\, T_A(y)\big),$$

$$I_{\overline{R_{NS}}(A)}(x) = sup_{y\in X} min\big(I_{R_{NS}}(x,y),\, I_A(y)\big),$$

$$F_{\overline{R_{NS}}(A)}(x) = sup_{y\in X} min\big(F_{R_{NS}}(x,y),\, F_A(y)\big).$$

Hence, single-valued NRSs (obtained by component-wise fuzzy-rough-style approximations of T, I, F) are precisely URSs of type M_{NS}.

Proof. We compute the lower approximation in $Dom(M_{NS})$ component-wise. For each $x \in X$,

$$\mu_{\underline{R_{NS}}(A)}(x) = \bigwedge_{y\in X}\big(N_{M_{NS}}(R_{NS}(x,y)) \;\oplus_{M_{NS}}\; \mu_A(y)\big).$$

Write

$$R_{NS}(x,y) = \big(T_{R_{NS}}(x,y), I_{R_{NS}}(x,y), F_{R_{NS}}(x,y)\big),$$

$$\mu_A(y) = (T_A(y), I_A(y), F_A(y)).$$

Then,

$$N_{M_{NS}}(R_{NS}(x,y)) = \big(1 - T_{R_{NS}}(x,y),\, 1 - I_{R_{NS}}(x,y),\, 1 - F_{R_{NS}}(x,y)\big),$$

and applying $\oplus_{M_{NS}}$ gives

$$N_{M_{NS}}(R_{NS}(x,y)) \;\oplus_{M_{NS}}\; \mu_A(y)$$

$$= \Big(max\big(1 - T_{R_{NS}}(x,y), T_A(y)\big), max\big(1 - I_{R_{NS}}(x,y), I_A(y)\big), max\big(1 - F_{R_{NS}}(x,y), F_A(y)\big)\Big).$$

Taking the infimum $\wedge$ over all $y \in X$ with respect to the component-wise order yields exactly the three formulas for $T_{\underline{R_{NS}}(A)}(x)$, $I_{\underline{R_{NS}}(A)}(x)$ and $F_{\underline{R_{NS}}(A)}(x)$ given in the statement.

The computation for the upper approximation is analogous. We have

$$\mu_{\overline{R_{NS}}(A)}(x) = \vee_{y \in X}\big(R_{NS}(x,y) \otimes_{M_{NS}} \mu_A(y)\big),$$

and

$$R_{NS}(x,y) \otimes_{M_{NS}} \mu_A(y) = \Big(min\big(T_{R_{NS}}(x,y), T_A(y)\big), min\big(I_{R_{NS}}(x,y), I_A(y)\big), min\big(F_{R_{NS}}(x,y), F_A(y)\big)\Big).$$

Taking the supremum $\vee$ component-wise over all $y \in X$ gives the stated expressions for $T_{\overline{R_{NS}}(A)}(x)$, $I_{\overline{R_{NS}}(A)}(x)$ and $F_{\overline{R_{NS}}(A)}(x)$.

Thus, the URS construction with $M = M_{NS}$ coincides with the usual component-wise fuzzy-rough generalisation of NRSs, proving that single-valued NRSs are exactly URSs of type M_{NS}.

Definition 3.27 (M-inclusion and pointwise order). For $A, B \in U_M(X)$, we write $A \subseteq_M B$ if

$$\mu_A(x) \leq_M \mu_B(x) \qquad \text{for all } x \in X.$$

For M-relations $R, S: X \times X \to Dom(M)$ we write $R \leq_M S$ if

$$R(x,y) \leq_M S(x,y) \qquad \text{for all } x, y \in X.$$

Definition 3.28 (Complement of a U-Set). For $A \in U_M(X)$, its (lattice) complement $A^c \in U_M(X)$ is defined by

$$\mu_{A^c}(x) := N_M(\mu_A(x)), \quad x \in X.$$

Lemma 3.29 (Well-definedness). For every $A \in U_M(X)$, the formulas

$$\mu_{\underline{R_M}(A)}(x) = \wedge_{y \in X}\Big(N_M\big(R_M(x,y)\big) \oplus_M \mu_A(y)\Big),$$

$$\mu_{\overline{R_M}(A)}(x) = \vee_{y \in X}(R_M(x,y) \otimes_M \mu_A(y))$$

define U-Sets $\underline{R_M}(A), \overline{R_M}(A) \in U_M(X)$.

Proof. For each fixed $x \in X$ and $y \in X$, we have $R_M(x,y) \in Dom(M)$, hence $N_M(R_M(x,y)) \in Dom(M)$, and therefore, $N_M(R_M(x,y)) \oplus_M \mu_A(y) \in Dom(M)$ and

$R_M(x,y) \otimes_M \mu_A(y) \in Dom(M)$. Since X is finite and $Dom(M)$ is a lattice, the finite meet $\bigwedge_{y\in X}(\cdots)$ and finite join $\bigvee_{y\in X}(\cdots)$ exist in $Dom(M)$, so both membership values lie in $Dom(M)$. Thus, $\underline{R_M}(A)$ and $\overline{R_M}(A)$ are well-defined elements of $U_M(X)$.

Theorem 3.30 (Monotonicity in the approximated U-Set). If $A, B \in U_M(X)$ satisfy $A \subseteq_M B$, then

$$\underline{R_M}(A) \subseteq_M \underline{R_M}(B), \qquad \overline{R_M}(A) \subseteq_M \overline{R_M}(B).$$

Proof. Assume $A \subseteq_M B$, i.e. $\mu_A(y) \leq_M \mu_B(y)$ for all $y \in X$. Fix $x \in X$. Since $\oplus_M$ and $\otimes_M$ are lattice join/meet, they are isotone in each argument. Hence, for every $y \in X$,

$$N_M(R_M(x,y)) \oplus_M \mu_A(y) \ \leq_M \ N_M(R_M(x,y)) \oplus_M \mu_B(y),$$

and

$$R_M(x,y) \otimes_M \mu_A(y) \ \leq_M \ R_M(x,y) \otimes_M \mu_B(y).$$

Taking the meet over $y \in X$ in the first inequality yields $\mu_{\underline{R_M}(A)}(x) \leq_M \mu_{\underline{R_M}(B)}(x)$. Taking the join over $y \in X$ in the second yields $\mu_{\overline{R_M}(A)}(x) \leq_M \mu_{\overline{R_M}(B)}(x)$. Since x was arbitrary, both inclusions follow.

Theorem 3.31 (Monotonicity in the relation). Let $R, S: X \times X \to Dom(M)$ be M-relations with $R \leq_M S$. Then, for every $A \in U_M(X)$, $\underline{S}(A) \subseteq_M \underline{R}(A)$, $\overline{R}(A) \subseteq_M \overline{S}(A)$, where $\underline{R}, \overline{R}$ denote the approximations induced by R (and similarly for S).

Proof. Assume $R \leq_M S$. Since N_M is order-reversing in a De Morgan lattice,

$$N_M\big(S(x,y)\big) \leq_M N_M\big(R(x,y)\big) \qquad \text{for all } x, y \in X.$$

Fix $x \in X$ and $y \in X$. By isotonicity of $\oplus_M$,

$$N_M(S(x,y)) \oplus_M \mu_A(y) \ \leq_M \ N_M(R(x,y)) \oplus_M \mu_A(y).$$

Taking the meet over y gives $\mu_{\underline{S}(A)}(x) \leq_M \mu_{\underline{R}(A)}(x)$, hence $\underline{S}(A) \subseteq_M \underline{R}(A)$.

For the upper approximation, isotonicity of $\otimes_M$ yields

$$R(x,y) \otimes_M \mu_A(y) \ \leq_M \ S(x,y) \otimes_M \mu_A(y),$$

and taking the join over y gives $\overline{R}(A) \subseteq_M \overline{S}(A)$.

For reference, a concise comparison of FRSs, U-Sets, and URSs is provided in Table 2.

Table 2. A concise comparison of fuzzy rough sets, uncertain sets, and uncertain rough sets.

Aspect	**Fuzzy rough sets (FRSs)**	**Uncertain sets (U-Sets)**	**Uncertain sough sets (URSs)**
Universe	finite U	nonempty X	finite X
Membership/ degree	fuzzy set $A: U \to [0,1]$	$\mu_A: X \to Dom(M) \subseteq [0,1]^k$	$\mu_A: X \to Dom(M) \subseteq [0,1]^k$
Underlying relation	fuzzy relation $R: U \times U \to [0,1]$	not required (standalone membership model)	M-relation $R_M: X \times X \to Dom(M)$
Algebraic structure on degrees	$([0,1], \leq, max, min, 1 - \cdot)$	$(Dom(M), \leq_M, \oplus_M, \otimes_M, N_M)$ (typically De Morgan lattice)	same as U-Sets (needed to define approximations)
Approximation operators	$\underline{R}(A)$, $\overline{R}(A)$ via $infmax$ and $supmin$	not part of the basic notion	$\underline{R_M}(A)$, $\overline{R_M}(A)$ via $\wedge(N_M(R_M) \oplus_M \mu_A)$ and $\vee(R_M \otimes_M \mu_A)$
Output form	pair of fuzzy sets (lower/upper)	single M-valued set	pair of M-valued sets (lower/upper)
Expressiveness/ scope	models graded indiscernibility with scalar degrees	unifies fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, etc.	unifies rough approximations across the same families; FRS is a special case
Typical use	feature selection, decision support, uncertainty-aware classification	unified operations/similarity across heterogeneous uncertainty models	approximation-based reasoning under general (vector/multi-degree) uncertainty

3.4. Functorial rough sets

In this subsection, a categorical (functorial) version of the URS framework is introduced. The construction combines the functorial viewpoint of functorial sets with the approximation operators of URSs.

Throughout, we fix:

- a small category C;
- a covariant functor $F: C \to Set$ (so (C, F) is a functorial set);
- an uncertainty model M whose degree domain $Dom(M)$ is equipped with a *bounded De Morgan lattice* structure

$$(Dom(M), \leq_M, \vee_M, \wedge_M, N_M, 0_M, 1_M),$$

where $\vee_M$ and $\wedge_M$ are the join and meet, N_M is an order-reversing involution (De Morgan negation), and $0_M, 1_M$ are the least and greatest elements. *Notation.* When convenient, we also write $\oplus_M := \vee_M$ and $\otimes_M := \wedge_M$, so that $\oplus_M$ (resp. $\otimes_M$) denotes the join (resp. meet) of this lattice.

As in the earlier sections, we assume (unless stated otherwise) that all universes under consideration are finite and nonempty; in particular, $F(X)$ is finite and nonempty for every $X \in Ob(C)$. Hence, all finite joins/meets that appear below are well-defined in $Dom(M)$.

For each object $X \in Ob(C)$, we write $F(X)$ for the underlying universe and denote by

$$U_M(F(X)) \ := \ \{\mu: F(X) \to Dom(M)\}$$

the set of all U-Sets of type M on $F(X)$.

Definition 3.38 (Pullback of U-Sets along morphisms). Let $f: X \to Y$ be a morphism in C. The induced map

$$F(f): F(X) \to F(Y)$$

defines a pullback (precomposition) operator

$$f^*: U_M(F(Y)) \to U_M(F(X)), \quad f^*(\mu) \ := \ \mu \circ F(f).$$

Equivalently, for $\mu \in U_M(F(Y))$ and $x \in F(X)$,

$$(f^*\mu)(x) = \mu(F(f)(x)).$$

This assignment is contravariantly functorial:

$$(id_X)^* = id_{U_M(F(X))}, \quad (g \circ f)^* = f^* \circ g^*$$

for all composable morphisms f, g.

Consequently, the rule

$$X \mapsto U_M\big(F(X)\big), \quad f \mapsto f^*$$

defines a functor

$$U_M \circ F: C^{op} \to Set.$$

We now define the functorial counterpart of rough approximations.

Definition 3.39 (Functorial rough structure and FunRS). Let C, F, and M be as above.

(i) A Functorial Rough Structure of type M on (C, F) consists of:

- for each object $X \in Ob(C)$, an M-relation

$$R_X: F(X) \times F(X) \to Dom(M),$$

together with the induced lower and upper approximation operators

$$\underline{R_X}, \ \overline{R_X}: U_M(F(X)) \to U_M(F(X)),$$

defined for $A \in U_M(F(X))$ and $x \in F(X)$ by

$$\mu_{\underline{R_X}(A)}(x) \;:=\; \wedge^M_{y\in F(X)}(N_M(R_X(x,y)) \;\vee_M\; \mu_A(y)),$$

$$\mu_{\overline{R_X}(A)}(x) \;:=\; \vee^M_{y\in F(X)}(R_X(x,y) \;\wedge_M\; \mu_A(y)),$$

where $\wedge^M$ and $\vee^M$ denote finite meet and finite join in $Dom(M)$.

- for every morphism $f: X \to Y$ in C and every $A \in U_M(F(Y))$, the following naturality conditions:

$$\underline{R_X}(f^*A) = f^*\left(\underline{R_Y}(A)\right), \quad \overline{R_X}(f^*A) = f^*\left(\overline{R_Y}(A)\right).$$

Equivalently, the families $(\underline{R_X})_X$ and $(\overline{R_X})_X$ form natural transformations

$$\underline{R}, \overline{R}: U_M \circ F \Rightarrow U_M \circ F.$$

(ii) A FunRS of type M over (C, F) is a choice of a Functorial Rough Structure $(R_X)_{X\in Ob(C)}$ together with a family of U-Sets

$$A_X \in U_M(F(X)), \qquad X \in Ob(C),$$

such that for every morphism $f: X \to Y$ in C one has

$$A_X = f^*(A_Y).$$

For each object X, the pair $\left(\underline{R_X}(A_X), \overline{R_X}(A_X)\right)$ is called the rough approximation of A at X.

Intuitively, a Functorial Rough Structure provides a coherent family of rough approximations across all objects of the category, and the naturality conditions ensure compatibility under change of universe along morphisms in C.

It is now shown that FunRSs generalise both URSs and functorial sets.

Theorem 3.40 (FunRSs generalise URSs). Let X_0 be a finite nonempty set and let $R_{M,0}: X_0 \times X_0 \to Dom(M)$ be an M −relation. Let C_0 be the category with a single object $*$ and only the identity morphism id_*, and let $F_0: C_0 \to Set$ be given by $F_0(*) = X_0$.

Then, FunRSs of type M over (C_0, F_0) are in one-to-one correspondence with URSs of type M on the universe X_0 with respect to the relation $R_{M,0}$.

Proof. Since C_0 has only one object and only id_*, the pullback $(id_*)^*$ is the identity on $U_M(X_0)$, so the naturality conditions in Definition 3.39 are vacuous. Thus, specifying a Functorial Rough Structure over (C_0, F_0) is exactly specifying a single M-relation on X_0, and the induced operators coincide with the URS operators on X_0. Likewise, a compatible family $(A_X)_X$ reduces to a single U-Set $A_* \in U_M(X_0)$.

Theorem 3.41 (FunRSs generalise functorial sets). Let (C, F) be a functorial set and let M be as above. Define for each object $X \in Ob(C)$ the identity M-relation on $F(X)$ by $R_X^{id}(x, y) := \{1_M, x = y, 0_M, x \neq y,\ x, y \in F(X)$. Then

(a) For every X and every $A \in U_M(F(X))$,

$$\underline{R_X^{id}}(A) = A, \qquad \overline{R_X^{id}}(A) = A.$$

(b) The families $(\underline{R_X^{id}})_X$ and $(\overline{R_X^{id}})_X$ satisfy the naturality conditions of Definition 3.39.

Consequently, every functorial set (C, F) can be viewed as a degenerate Functorial Rough Structure by equipping it with the relations R_X^{id}, and forgetting the rough-approximation data recovers the original functor F.

Proof.

(a) Fix $X \in Ob(C)$ and $A \in U_M(F(X))$. For any $x \in F(X)$,

$$\mu_{\underline{R_X^{id}}(A)}(x) = \wedge_{y \in F(X)}^{M}\big(N_M(R_X^{id}(x, y)) \vee_M \mu_A(y)\big).$$

If $y = x$, then $R_X^{id}(x, x) = 1_M$ so $N_M(1_M) = 0_M$, hence the corresponding term equals $0_M \vee_M \mu_A(x) = \mu_A(x)$. If $y \neq x$, then $R_X^{id}(x, y) = 0_M$ so $N_M(0_M) = 1_M$, hence the term equals $1_M \vee_M \mu_A(y) = 1_M$. Therefore, the meet is taken over $\mu_A(x)$ and several copies of 1_M, giving $\mu_{\underline{R_X^{id}}(A)}(x) = \mu_A(x)$.

Similarly,

$$\mu_{\overline{R_X^{id}}(A)}(x) = \vee_{y \in F(X)}^{M}\big(R_X^{id}(x, y) \wedge_M \mu_A(y)\big),$$

and the term for $y = x$ equals $1_M \wedge_M \mu_A(x) = \mu_A(x)$ while for $y \neq x$, it equals $0_M \wedge_M \mu_A(y) = 0_M$. Hence, the join is $\mu_A(x)$, so $\overline{R_X^{id}}(A) = A$.

(b) Since $\underline{R_X^{id}} = id_{U_M(F(X))}$ and $\overline{R_X^{id}} = id_{U_M(F(X))}$ for all X, the naturality conditions

$$\underline{R_X^{id}}(f^*A) = f^*\left(\underline{R_Y^{id}}(A)\right), \quad \overline{R_X^{id}}(f^*A) = f^*(\overline{R_Y^{id}}(A))$$

hold trivially for every morphism $f: X \to Y$.

Lemma 3.42 (Pullback preserves complements). Let $f: X \to Y$ be a morphism in C and $A \in U_M(F(Y))$. Then, $f^*(A^c) = (f^*A)^c$.

Proof. For $x \in F(X)$, by definition of pullback and complement,

$$\mu_{f^*(A^c)}(x) = \mu_{A^c}(F(f)(x)) = N_M\big(\mu_A(F(f)(x))\big) = N_M\big(\mu_{f^*A}(x)\big) = \mu_{(f^*A)^c}(x).$$

Hence, $f^*(A^c) = (f^*A)^c$.

Theorem 3.43 (Monotonicity of lower and upper approximations). Let $X \in Ob(C)$ and let $A, B \in U_M(F(X))$. If $A \subseteq_M B$ (i.e. $\mu_A(x) \leq_M \mu_B(x)$ for all x), then

$$\underline{R_X}(A) \subseteq_M \underline{R_X}(B), \quad \overline{R_X}(A) \subseteq_M \overline{R_X}(B).$$

Proof. Fix $x \in F(X)$. Since $\vee_M$ and $\wedge_M$ are order-preserving in each argument, for every $y \in F(X)$, we have

$$N_M(R_X(x,y)) \vee_M \mu_A(y) \ \leq_M \ N_M(R_X(x,y)) \vee_M \mu_B(y),$$

and

$$R_X(x,y) \wedge_M \mu_A(y) \ \leq_M \ R_X(x,y) \wedge_M \mu_B(y).$$

Taking the finite meet over y in the first inequality yields $\mu_{\underline{R_X}(A)}(x) \leq_M \mu_{\underline{R_X}(B)}(x)$, and taking the finite join over y in the second yields $\mu_{\overline{R_X}(A)}(x) \leq_M \mu_{\overline{R_X}(B)}(x)$. Since x was arbitrary, the claimed inclusions follow.

Theorem 3.44 (Sandwich property under reflexivity). Fix $X \in Ob(C)$. Assume the M-relation R_X is reflexive, i.e.

$$R_X(x,x) = 1_M \text{ for all } x \in F(X).$$

Then, for every $A \in U_M(F(X))$,

$$\underline{R_X}(A) \subseteq_M A \subseteq_M \overline{R_X}(A).$$

Proof. Fix $x \in F(X)$.

Lower $\leq$ original. In the definition of $\underline{R_X}(A)$, the $y = x$ term equals

$$N_M(R_X(x,x)) \vee_M \mu_A(x) = N_M(1_M) \vee_M \mu_A(x) = 0_M \vee_M \mu_A(x) = \mu_A(x).$$

Since a meet is $\leq_M$ each of its terms, we obtain $\mu_{\underline{R_X}(A)}(x) \leq_M \mu_A(x)$.

Original $\leq$ upper. In the definition of $\overline{R_X}(A)$, the $y = x$ term equals

$$R_X(x,x) \wedge_M \mu_A(x) = 1_M \wedge_M \mu_A(x) = \mu_A(x).$$

Since a join is $\geq_M$ each of its terms, we obtain $\mu_A(x) \leq_M \mu_{\overline{R_X}(A)}(x)$.

Combining both inequalities for each x proves the claim.

4. Conclusions

In this article, operations, similarity measures, and rough-set structures within the framework of U-Sets and closely related generalisations were examined. Concretely, basic set-theoretic operations (inclusion, complement, union/intersection, difference, and α-cuts) for U-Sets were formalised under a general De Morgan lattice of uncertainty degrees, and axiomatic uncertain similarity measures were introduced together with representative L_1- and Chebyshev-type constructions. The author also defined URSs induced by M-valued relations and established that the resulting lower/upper approximation operators recover classical FRSs and single-valued NRSs as special cases. Finally, a functorial extension was outlined in which rough approximations are required to be natural with respect to morphisms, thereby enabling the coherent transport of approximations across changing universes.

It is expected that future research will further clarify how the concepts developed here behave in the settings of Uncertain Graphs, Uncertain HyperGraphs, and Uncertain SuperHyperGraphs, and how the proposed operations and similarity measures can be leveraged for structure-aware comparison, clustering, and decision-making on such uncertain relational models [26, 27].

ACKNOWLEDGMENTS

Sincere thanks are extended to all those who provided insights, great encouragement, and practical support during the course of this research work. Appreciation is also expressed to the readers and the scholars whose publications are cited herein, as their prior contributions provide essential background for the present study.

STATEMENT ON THE USE OF GENERATIVE AI

Generative AI and other AI-assisted tools were used only for limited support activities, such as improving English grammar and clarity, assisting with translation, and troubleshooting LaTeX compilation errors. These tools were applied in a manner consistent with accepted research-ethics standards. All final content and conclusions were subsequently reviewed and verified by the author.

COMPETING INTERESTS

The author declares that there is no conflict of interests regarding the publication of this article.

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